Does modal theory need to be Introduced to Intuitionistic, Epistemic, and Conditional Logic, or have they already met?

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1. The original impetus for the theory of combining logics by Dov Gabbay [1]("Fibred Semantics and the Weaving of Logics Part I: Modal and Intuitionistic Logics", JSL, 1996, pp.1057-1120) began with the problem of combining Modal and Intuitionistic logical systems. The problem was, as he initially expressed it, to introduce modality into Intuitionistic logic. The range of those logics which can be combined has since been extended beyond those systems originally considered, and would now probably include the introduction of modals into counterfactual and epistemic logics as well. The presumption, however, at least at the beginning was that the problem was to introduce modals into a system in where they were absent.

In the present study I want to return to a few considerations about the original case for combining modal and Intuitionistic logics, and to also consider examples of the combination of modal with some conditional logics. These considerations are prompted by a structuralist view of logic. I do not believe that these observations affect the formal results of Gabbay that are already in place, but they do make us think about what we want to achieve by combination. The reason is that on a structuralist view of the matter, there are many significant modals already present in Intuitionistic,

[1] "*Fibred Semantics and the Weaving of Logics Part I: Modal and Intuitionistic Logics*", *Journal of Symbolic Logic*, 1996, pp.1057-1120.

counterfacual and epistemic logics, - only that fact is not readily apparent if we attend only to the syntactic presence of a familiar indicator of necessity or possibility.

To give some detailed account of what we have in mind, we turn first to an explanation of what we mean by a structuralist account of the modal operators.

2. ***Structuralist Modals***. This idea of modals, structuralist –style, is simply explained. First one introduces the notion of an implication structure, $\mathfrak{I} = \langle S, \Rightarrow \rangle$, by which we mean any non-empty set S together with an implication relation, that is, any relation on S that satisfies the following six (redundant) conditions:

1. **Reflexivity**. $A \Rightarrow A$, for all A in S.
2. **Projection**. $A_1, \ldots, A_n \Rightarrow A_k$ for $1 \leq k \leq n$.
3. **Simplification**. If $A_1, A_1, \ldots, A_n \Rightarrow B$, then $A_1, \ldots, A_n \Rightarrow B$.
4. **Permutation**. If $A_1, \ldots, A_n \Rightarrow B$, then $A_{f(1)}, \ldots, A_{f(n)} \Rightarrow B$ for any permutation f of $\{1, 2, \ldots, n\}$.
5. **Dilution**. If $A_1, \ldots, A_n \Rightarrow B$, then $A_1, \ldots, A_n, C \Rightarrow B$.
6. **Cut.** If $A_1, \ldots, A_n \Rightarrow B$, and $B, B_1, \ldots, B_m \Rightarrow C$, then $A_1, \ldots, A_n, B_1, \ldots, B_m \Rightarrow C$.

A general definition of each of the logical operators, including quantification can be given for implication structures. They are defined as special kinds of functions that map elements or pairs of elements of the set S to S. That study has been given in detail elsewhere, and is not an issue here.[2] The modal operators are defined relative to the implication relations on implication structures as a special kind of function that maps the members of S to S.

[2] A. Koslow, *A Structuralist theory of Logic*, Camprigde University Press, N.Y., 1992.

More exactly, a structuralist theory of modality requires that any modal operator φ on an implication structure is function φ that maps the set S to itself, and satisfies the following two simple conditions:

(**M1**): (1) If $A_1, \ldots, A_n \Rightarrow B$, then $\varphi(A_1), \ldots, \varphi(A_n) \Rightarrow \varphi(B)$, *and*
(**M2**): There are A and B in S, such that $\varphi(A \vee B) \Rightarrow \varphi(A) \vee \varphi(B)$ *fails.*

The account is structuralist simply because it describes this special kind of operator by reference only to the implication relation on various implication structures. It is an account that includes most if not all the familiar normal modals and some of the non-normal ones as well; it enables one to prove that all modal operators are non-extensional, and it provides exact proofs for the familiar Kripke results on accessibility relations, and does so without using truth-conditions. It does this by relying only on implication. It has wide scope and many applications that speak in its favor. Since so much of what follows depends critically on our account of structural modals, it is worthwhile looking at it from at least two different perspectives: what would happen say if the second condition failed, and how does it compare with say the specific notions of necessity and possibility as embedded in Aristotle's Modal Square of Opposition.

3. ***Structural Modals, the Failure of (M2), and Contingency.*** First, a caveat. Our account of modal operators requires a condition (M1) of what happens when there are multiple premises. That is important, since the difference between necessity modals and those of possibility lies in the behavior when there are many items implying another; not just one. Also the first condition tells us something important about modals: they preserve implications. For that reason alone modal logic is an integral part of any study of even the non-modal logics: modals with respect to implication are invariants of that relation. Let us then assume that the first condition is in place, and leave to one side the possible variations on that condition.

The second condition of modality (M2) says that there is some disjunction of members of an implication structure, such that the modal does not distribute over that disjunction. There is a way of expressing this second condition even when there may not be a disjunction in the structure (the official condition). It involves the use of dual implication relations but we do not need that more general account here.

(**M2**) is very plausible. For example we normally take it that there are disjunctions $A \vee B$ which are necessary, where neither A nor B is necessary. And, knowledge would be modal (assuming that knowledge satisfies (**M1**), which is moot) , for surely it is the case that some disjunction is known, but neither of the disjuncts is known. There are many other cases of equal interest that turn out to be modal. That is evidence of a sort that the condition is onto something. If we consider those cases where the first condition holds, but the second condition fails, then a number of dramatic undesirable consequences follow. Here are several that are closely related:

(1) If there is a binary accessibility relation R(u,v) associated with a modal operator $\Box$, such that such that the usual clauses about necessarily P being true in a world if and only if it is true in all worlds accessible from it (and the usual dual clause for P being possible in a world if and only if it is true is some world accessible from it), then (M2) fails if and only if R is a function (that is, for all worlds u, v, and w, if R(w, u) and R((w, v), then u = v..Equivalently,

(2) (M2) holds if and only if R branches at some world.

However the most obvious consequence is that if the second condition fails, and the modal $\Box$ distributes over all disjunctions, then given the additional assumptions that necessity implies possibility, $\Box(A) \Rightarrow \Diamond(A)$ for all A, and that $\Box(C)$ is a theorem for some C, then

(3) $\Diamond(A) \Rightarrow \Box(A)$, for all A. That is, possibility implies necessity.

So there is the collapse of necessity with possibility. Even more striking, is this consequence:

(4) There are no contingent statements.

The reason is simple: let us say that P is *contingent* if and only if $\Diamond(P) \wedge \Diamond(\neg P)$ is consistent. It follows that $\Diamond(P) \wedge \Diamond(\neg P)$ is inconsistent, because, from (3) we have that $\Diamond(P)$ implies $\Box(P)$, and (from the usual classical assumption that $\Box(P)$ implies $\neg \Diamond(\neg P)$, we have that $\Diamond(\neg P)$ implies $\neg\Box(P)$. Therefore $\Diamond(P) \wedge \Diamond(\neg P)$ implies the contradiction $\Box(P) \wedge \neg\Box(P)$.

In other words, the second condition of modality requires that there is at least one member of the implication structure that is contingent.

4. ***Structural Modals, The Modal Square of Opposition, and Contingency***. One easy comparison is to what is now called Aristotle's Modal Square of Opposition.[3] It provides a schema which sets forth the relations which the special modals of necessity and possibility are supposed to have to each other. Let P stand for any proposition or statement, and "$\Box(P)$", and "$\Diamond(P)$" stand for "It is necessary that P", and "It is possible that P" respectively. The Modal Square then gets represented this way:

$\Box(P)$ ● ● $\Box(\neg P)$

$\Diamond(P)$ ● ● $\Diamond(\neg P)$

The statements along the diagonal are supposed to be *contradictories*: they are inconsistent with each other, and so too are their negations. The statements in the upper corners imply the corresponding statements in the lower corners. The statements in the two upper corners are required to be *contraries*: they are inconsistent with each

[3] Aristotle, *De Interpretatione* 21b10ff and *Prior Analytics* 32a18-28, Larry Horn "*Contradiction"* (,2006), in *The Stanford Encyclopedia of Philosophy*, and W.&M.Kneale, *The Development of Logic*, Oxford University Press (1962), 85ff.

other, and their negations are consistent with each other (not inconsistent). Lastly, the statements in the two lower corners are *subcontraries*: they are consistent with each other, and their negations are inconsistent with each other.

Usually these relations between the various necessities and possibilities are stated using some unexplained notion of truth and two special modals (contraries are statements that *cannot be true* together, and whose negations *can be true* together. I have phrased things in terms of consistency and inconsistency, so as to avoid an unanalyzed notion of truth, and the use of two other unexplained modals ("can" and "cannot").

There are some obvious conclusions we can draw. Since for any statement P, "It is possible that P", and "It is necessary that not P" are inconsistent with each other, it follows that necessity implies not possibly not. And from the condition that their negations are also inconsistent with each other we can conclude that for any P,

$$(4)\ \Diamond(P) \Leftrightarrow \neg\Box(\neg P).$$

Similarly, from the inconsistency of "It is necessary that P" with "It is possible that $\neg$P", and the inconsistency of their negations with each other, we can conclude that

$$(5)\ \Box(P) \Leftrightarrow \neg\Diamond(\neg P).$$

These relations are usually endorsed in standard modal theory that is classical, though of course not Intuitionistic modal theory.

If we turn next to the two upper corners, we conclude that since they are contraries, they are inconsistent with each other, then $\Box(P)$ implies the negation of $\Box(\neg P)$, so that by (4), it follows that

$$(6)\ \Box(P) \Rightarrow \Diamond(P), \text{ for all P.}$$

This result of Aristotle is of course familiar, but limited to a special class of modals (It will of course always hold in all classical modal systems which extend the modal system T, but it doesn't hold for the familiar modal system of Gödel-Löb). Our structural account gets it right for the special as well as the general cases.

The real surprise of Aristotelian modals comes with the subcontraries of the modal Square. Subcontraries are consistent with each other and their negations are inconsistent with each other. The latter condition yields the familiar neccessity-implies-possibility condition. But the first condition is surprising for it requires that for every P, the possibility of P and the possibility of not P are consistent with each other. That is, it is required by the Modal Square that for all P, it is consistent that $\Diamond(P) \wedge \Diamond(\neg P)$. This result also follows from the claim that negations of the statements in the two upper corners are consistent with each other. Consequently, on the Aristotelian view

All statements are contingent.

(at least for all those statements that satisfy the Modal Square).This is not good news! It suggests that there will be a problem with those statements that attribute necessity to statements, that is, statements of the form $\Box(P)$. The reason is that $\Box(P)$ implies $\neg\Diamond(\neg P)$, which in turn implies $\neg[\Diamond(P) \wedge \Diamond(\neg P)]$. Suppose now that P is some consistent sentence such that $\Box(P)$ is true in all possible worlds. Then $[\Diamond(P) \wedge \Diamond(\neg P)]$ is not true in any possible world –that is, it can't be true; it's equivalent to a contradiction. So P is not contingent. But according to Aristotle, all statements that fit the Modal Square are contingent. Therefore on Aristotle's account of necessity and possibility, for every consistent P, "$\Box(P)$" is false in every possible world. Of course, even if all statements of necessity are inconsistent, the formal relations between the four types of statements still hold. However it's not ideal to have a theory of necessity and possibility when every statement of necessity is false. Posssible defects of the modal square of opposition are not the target of the present study. Much more interesting is the claim that all statements are contingent, as a mark of (Aristotelian) modality.[4]

It is now relatively easy to situate the structural account of modal operators with respect to the well known Aristotelian one. We have already remarked on how the

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[4] Of course it is entirely possible that the explanation of subcontraries, and the notions of necessity and posssibility have not been translated properly, and the fault lies with our translations rather than with Aristotle.

relations required by him are only a very limited part of the story that has been incorporated into modern theory as special cases. His claim is that the proper theory for modals like necessity and possibility requires that every statement or proposition is contingent. The structural theory of modals on the other hand is a more modest and more reasonable position: Our second condition turns out to be equivalent to the condition that there is *at least one member* of the implication structure that is contingent.

5. *Introducing the Modals*. Dov Gabbay in his seminal paper on fibring, introduced the problem of introducing the modals into contexts where they presumably are not already present. It is worth recalling his wording. Of Intuitionistic modal logics he says

> "Such logics are abundant in the literature. The first syntactical attempt of introducing modality into intuitionistic logic was by Fitch[20]. Many others followed. The motivation was mainly philosophical in search for intuitionistic constructive modality, as both intuitionistic logic and modal logic have solid and independent philosophical reasons."

What is very interesting about this quote from the 1996 paper of Gabbay is that an appeal was made to the *syntactical* attempt of introducing modals into Intuitionistic theory. It is true that in the Intuitionistic sentential calculus, there are no terms like boxes or diamonds or their syntactically presented equivalents. Hence it made sense to speak of introducing them, where they weren't present on the face of it.

This observation is certainly true, but it rests I think on a simple attention to the surface grammar of the expressions of Intuitionistic logic. From a structuralist perspective, however, it is misleading. There are modals in Intuitionistic logic. In fact there are at least two, and if we are correct, there are infinitely many modals in that theory.

Before we turn to a discussion of some logics that apparently do not have some modal operators present, there is one caveat that we should make. One has to be careful about some tacit conditions which, when made explicit, may make for some trouble. Suppose we are Intuitionists who are concerned with a structure of mathematical statements and the logical relations between them. It seems fair to say that these mathematical statements have a necessity to them, even though that fact isn't usually made explicit. So let us say, as dedicated Intuitionists, that there is some necessity modal $\Box$ (some notion of mathematical necessity that is Intuitionistically acceptable) and that for every (mathematical) statement A there is some statement B such that A is equivalent to $\Box$(B). This would be one way of expressing the idea that every A is necessary. Now in our desire to add some modal theory to Intuitionism, we might add on some axioms for a simple modal system –say that necessity is both a T-modal, and a K4 modal (so $\Box$(A) implies A, and $\Box$(A) implies $\Box\Box$(A). If the Intuitionist then expresses the requirement that mathematical statements are necessary, using "$\Box$", then there is serious trouble. For it is simple to see that imposing the T and K4 condition on modals, as well as the condition that all the statements are necessary, guarantees total modal collapse: $\Box$(A) is equivalent to A for every A.. Another way of putting the matter is this: If we introduce a modal that is T and K4, into the Intuitionistic Sentential Calculus (ISC), then that modal will be useless for expressing the deep thesis that every mathematical statement (that is Intuitionistically acceptable of course) is necessary. Why then should such an introduction into Intuitionistic theory be welcome? This is a special case of a general result:

> If a modal operator is a mapping of the set S of an implication structure onto S, then that modal operator cannot be both a T and a K4 modal.[5]

I want now to turn to the consideration of some logical theories, where we might want to introduce modal operators, where it *seems* they are lacking. We shall look at a few logical theories that do not at first sight look like modal logics, and we shall argue that despite appearances to the contrary, some of these logics do possess interesting modal operators, and some do not. Consequently, one could wind up introducing a modal operator where indeed there was none, or one might be introducing a modal operator where there are already some present, and the interaction of these modals would require careful scrutiny. No one would wish to introduce a new modal into an accepted system, if the result of that introduction might result in a system that was not acceptable. We turn next to a consideration of a few logical systems where modal operators are present but not evident, and those modals cannot be discounted, since they have a role in explaining some key features of these systems.

The examples that we have in mind are the Classical Sentential Calculus (CSC), the Intuitionistic Sentential Calculus (ISC), D. Lewis' favorite version of the logic of counterfactuals (VC), and a theory of epistemic conditionals due to Horacio Arlo-Costa (which provides and studies a formalization of the belief revision theory of Isaac Levi).

6. *The Classical Sentential Calculus*. For the simple case of the Classical Sentential Calculus we shall use the associated implication structure $< L_{CSC}, \Rightarrow^{CSC} >$ consisting of

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[5] The special application of this result to Intuitionism was brought to my attention by Charles Parsons.

the Language L_{CSC} together with some implication relation $\Rightarrow^{CSC}$, on it that is either semantic or proof-theoretic, it doesn't matter which.

Consider the possibility of there being a modal operator associated with the material conditional. Everyone knows that there is a difference between the material conditional of Classical Sentential Logic, and say counterfactual conditionals. The easy way to show that is by way of particular examples. At a more satisfying level one illustrates the fact that the material conditional is extensional, and the counterfactual conditional is not. That is easy enough. But there is another way of marking the difference, in a way that explains why there is this difference in extensionality. For any material conditional $A \supset B$, rewrite it using the prefix expression " $A \supset$ " ("If A then"). That is. define a function φ_A which maps sentences of (CSC) to sentences of (CSC) this way: to each sentence B, $\varphi_A(B)$ is $A \supset B$. That is, the function indexed to A assigns to any B the result of prefixing "If A, then" to the sentence B.

The next question is a simple one. Is φ_A a structural modal? Are the two conditions satisfied. It is easy to see that the first condition (M1) holds. But the second (M2), does not: consider any A, B, and C. Then $\varphi_A(B \vee C)$ implies $\varphi_A(B) \vee \varphi_A(C)$. The reason is that $A \supset (B \vee C)$ implies $(A \supset B) \vee (A \supset C)$, in (CSC). Let us call the operator defined with the aid of the initial prefix of the material conditional, the *conditional antecedent operator*. Then it is clear that the conditional antecedent operator for the material conditional is not a (structural) modal operator.

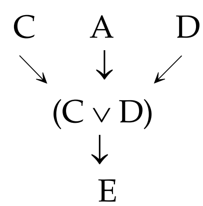
If we form the similar construction for the counterfactual conditional "If A were the case, then B would be the case" ($A \Box\!\!\rightarrow B$), by using the prefix "$A \Box\!\!\rightarrow$" (If A were the case, then"), then as we shall prove below, the conditional antecedent operator for counterfactual conditionals (ψ_A), which maps any B to A □→B will generally be a modal operator. So in the case of counterfactual conditionals, we will have shown that A□→B has the form ψ_A (B) where ψ_A is a structural modal. Since all structural modals are provably non-extensional, we have as it were located the source of the extensionality of counterfactuals. Some philosophers hold that indicative conditionals have as their truth-conditions exactly those for the material conditional (Grice for one), but that counterfactuals are another matter. We now can see that a nice way of marking the distinction between material and counterfactual conditionals is that counterfactual conditionals are modal, and material conditionals are not. More exactly, the material conditional antecedent operator is not a modal operator, but the counterfactual conditional antecedent operator is a modal operator.

7. ***The Intuitionistic Sentential Calculus***. The second kind of theory I have in mind is the Intuitionistic Sentential Calclulus (ISC). Consider the implication structure $\mathfrak{I}$ = <ISC, $\Rightarrow$ > consisting of the set of sentences of (ISC) of the Intuitionistic sentential calculus, and let the implication relation $A_1, A_2, \ldots, A_n \Rightarrow B$ be defined as holding if and only if $[A_1 \wedge A_2 \wedge \ldots \wedge A_n \rightarrow B]$ is a theorem of Intuitionistic Sentential Calculus.

For any A and B, consider the Intuitionistic conditional $A \rightarrow B$, and define φ_A as the function which maps each member C of the implication structure to the sentence $A \rightarrow C$. (Here "$\rightarrow$" means the Intuitionistic, not the material conditional. This means that "$A \rightarrow C$" is not equivalent to "$\neg A \vee B$"). Our claim is that the operator φ_A is a modal operator on the structure. We shall call such operators *Intuitionistic condi-*

tional antecedent modal operators when they are modal operators.

To see this we have to check the two conditions of modality. (M1) requires that for any $A_1,\ldots, A_n$ and B, (1) If $A_1,\ldots, A_n \Rightarrow$ B, then $\varphi_A(A_1),\ldots, \varphi_A(A_n) \Rightarrow \varphi_A(B)$, and M2 requires that (2) There are sentences C, D, of (ISC), such that $\varphi_A(C \vee D) \Rightarrow \varphi_A(C) \vee \varphi_A(D)$ *fails.* It is easy enough to prove (1) in the Intuitionistic sentential calculus, and a simple finite structure suffices to show (2). Just take any A, B, C, D, and E such that A implies the disjunction of C and D, which in turn implies E, while C and D of course each imply their disjunction, but A neither implies nor is implied by C, and D neither implies nor is implied by A. That is,



Note that $A \rightarrow (C \vee D)$ is E which is a thesis of the structure (it is implied by all the elements of the structure), but $(A \rightarrow C) \vee (A \rightarrow D)$ is not a thesis of the structure (it is equivalent to $(C \vee D)$. So $A \rightarrow (C \vee D)$ cannot imply $(A \rightarrow C) \vee (A \rightarrow D)$ (anything implied by a thesis has to be a thesis).

We should note that this is not the only modal operator on the Intuitionistic implication structure. There is another that is associated with double negation. The function φ_N which maps each element C of the Intuitionistic implication structure to its double negation ($\neg\neg$ C), is also a structural modal operator. However the Intuitionistic modal associated with the conditional is of special interest at present.

We have described a modal φ_A operator that to each C in the structure assigns the conditional ($A \rightarrow C$). These conditional modals are indexed to elements of the structure, and therefore there may be infinitley many such Intuitionistic modals on the structure. We shall say that such a structure is an example of a *modal field* – that is an implication structure which to each element A of the structure there is assigned an operator φ_A - usually different operators for different members of the structure, and such that, except for certain elements of the structure, these operators are modal operators. The exceptions, - those As for which the operator -φ_A is not a modal operator will be called *singularities* of the modal field.

We note that in the Intuitionistic case there are at least two singularities: if A is a thesis of the structure (every member of the structure implies it) and if A is contradictory (it implies every member of the structure), then the corresponding operator will not be modal. The reason is that if A is a thesis, then ($A \rightarrow C$) is equivalent to C, so that the second condition of modality (M2) fails. And if A is a contradiction then ($A \rightarrow C$) is a thesis, and again the second condition of modality fails.

Let us say that the modal φ implies the modal ψ if and only if $\varphi(C) \Rightarrow \psi(C)$ for all C in the implication structure, (and they are equivalent if and only if the converse holds) then for all the Intuitionistic conditional antecedent modals

φ_A implies φ_B if and only if B implies A..

So we have the result that all the Intuitionistic conditional antecedent modals are independent of each other (neither one implies the other) if and only of they are indexed to elements of the structure that are independent of each other. And that means that generally, there are a lot of them.

It is worth noting that the Intuitionistic conditional antecedent modals φ_A are unusual as modals go. They seem to combine features of both necessity and possibility. Thus (1) they distribute over implications with multiple antecedents (M1), (2) for all modals φ_A we have for all B, that $B \Rightarrow \varphi_A(B)$, which is characteristic of possibility rather than necessity, (3) $\varphi_A(B) \Leftrightarrow \varphi_A\varphi_A(B)$ for all B which is shared by some necessity and possibility modals, and (4) the necessitation rule holds: if B is a thesis, then so too is $\varphi_A(B)$.

This is an interesting, rare kind of modal that Intuitionism provides which doesn't get taken into account when the combinations of modal and Intuitionistic logics are considered. We think that it is not only an interesting modal, but that it is an important one. It explains, for example, why the conditional of Intuitionistic logic is non-extensional. The reason is that any Intuitionistic conditional $A \to B$ is representable as $\varphi_A(B)$. Therefore, if φ_A is modal operator, that is enough to guarantee the non-extensionality of $\varphi_A(B)$ – that is, $A \to B$. It also explains why the Intuitionistic conditional differs from the classical material conditional. We know that $\neg A \vee B$ implies the Intuitionistic $A \to B$. Suppose that the converse also held so that $A \to B$ is equivalent (in (ISC) to $\neg A \vee B$. If that is so, then the operator which assigns to every C, the conditional $A \to C$ would no longer be a modal operator, because the operator would then distribute over disjunctions: $A \to (B \vee C)$ implies $\neg A \vee (B \vee C)$ which implies $(\neg A \vee B) \vee (\neg A \vee C)$, for every A,B, and

C. Therefore $\varphi_A(B \vee C)$ implies $\varphi_A(B) \vee \varphi_A(C)$, so that the second condition of structural modality fails. Moral: you can't have any Intuitionistic conditional $(A \rightarrow B)$ imply the material conditional $(\neg A \vee B)$, without the Intuitionistic antecedent conditional operator φ_A losing it's modal status.

It seems to me that these are important modals of Intuitionistic logic. Therefore, before we introduce various modal operators into Intuitionistic logics, we ought to check first whether those modal operators of aren't of a type that is already present, and check also whether the introduction of specific modals that are introduced do not interact with others already there in such a way as to result in an Intuitionistically unacceptable system. That would be like injecting an unfriendly virus into the host system.

The Intuitionistic conditional antecedent modals are an example of a modal field. There are others. There are other examples of modal fields in the philosophy of science and epistemology. We close with a brief discussion of two other examples of special interest: counterfactual conditionals, and a class of epistemic conditionals discussed by Horacio Arlo-Costa that are important for belief revision. Each is an example of a presumed candidate for combining with a modal logic. Each however is yet another important type of logical system that, despite appearances to the contrary, already possess significant modals of their own.

8. *Counterfactual Conditionals and their Modals*. We shall use D. Lewis' axiomatization of his system VC[6] which is an extension of the classical sentential calculus, with the added connective (box-arrow) "$\Box\!\!\rightarrow$", where "$A \Box\!\!\rightarrow B$" is read as a would counterfactual

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[6] D. Lewis, *Counterfactuals*, Harvard University Press, Cambridge, 1973, pp.132-33.

"If A were the case then it would be the case that B" and the dual "$\Diamond\!\!\rightarrow$" is read as a might counterfactual "If A were the case then B might be the case" ($A \Diamond\!\!\rightarrow B$). In addition to the rule of Modus Ponens we have the rule for distribution over implications:

(**D**) *For any* C, if $A_1,\ldots,A_n$ ==> B, *then* $C \Box\!\!\rightarrow A_1, \ldots, C \Box\!\!\rightarrow A_n$ ==> $C \Box\!\!\rightarrow B$.

The following are Lewis'axioms for **VC**:

(1) All truth-functional tautologies of the classical sentential calculus, and for all A, B, and C,

(2) $A \Box\!\!\rightarrow A$

(3) $(\neg A \Box\!\!\rightarrow A) \Rightarrow B \Box\!\!\rightarrow A$

(4) $\neg(A \Box\!\!\rightarrow \neg A) \Rightarrow [(A \,\&\, B) \Box\!\!\rightarrow C$ <--> $A \Box\!\!\rightarrow (B$ --> $C)]$

(5) $A \Box\!\!\rightarrow B \Rightarrow (A$ --> $B)$

(6) $(A \,\&\, B) \Rightarrow A \Box\!\!\rightarrow B$.

One should also add the requirement that the replacement of logically equivalent expressions in any formula will result in logically equivalent expressions.

Using the deducibility relation on the set of sentences of **VC**, we shall say that $A_1,\ldots, A_n \Rightarrow B$ if and only if $A_1,\ldots, A_n$ --> B is a theorem of **VC**. Let VC be the language of the system **VC**, and let the implication relation be given by "$\Rightarrow$" as we have just defined it. Thus we can form a counterfactual implicational structure $\mathfrak{I} = \langle VC, \Rightarrow \rangle$. Let $\square_A$ be a function which assigns to every C in the implication structure the sentence $A \Box\!\!\rightarrow C$. It follows from rule (**D**) that φ_A distributes over implication. That is, the first condition

(**M1**) of structural modality is satisfied. The requirement of C*onditional Excluded Middle* says that

(**CEM**): *For every* A *and* B, *it is a theorem of* **VC** *that* $[A \square\!\!\rightarrow B \vee A \square\!\!\rightarrow \neg B]$.

We note that (**CEM**) is not a theorem of **VC.** Let us say that A satisfies CEM if and only if for all B, $[A \square\!\!\rightarrow B \vee A \square\!\!\rightarrow \neg B]$ is a theorem of VC. Then there is a simple result about when the operator $\square_A$ is a modal operator:

$\square_A$ is a modal operator on the structure if and only if A fails to satisfy (**CEM**).

The proof is relatively simple. Suppose that A satisfies (**CEM**). It follows that $\square_A(C) \vee \square_A(\neg C)$ is a theorem for every C. Therefore $\neg\square_A \neg C) \Rightarrow \square_A (C)$ for every C. Then note that for any B and C, $\square_A (B \vee C) \Leftrightarrow \square_A(\neg B \rightarrow C) \Rightarrow \square_A (\neg B) \rightarrow \square_A (C) \Rightarrow \neg\square_A \neg B) \vee \square_A(C) \Rightarrow \square_A(B) \vee \square_A(C)$. Thus If A satisfies **CEM**), then φ_A is not a modal. For the other half of the result, suppose that A fails to satisfy (**CEM**). Then $\square_A(C) \vee \square_A(\neg C)$ is not a theorem for some C. Now note that $\square_A(C \vee \neg C)$ is a theorem for every C. The reason is that $A \Rightarrow (C \vee \neg C)$ for every A, so by Rule (D), we have $\square_A (A) \Rightarrow \square_A (C \vee \neg C)$. Since $\square_A (A)$ is a theorem (Axiom (2)), it follows that $\square_A (C \vee \neg C)$ is also a theorem. Consequently, since $\square_A C \vee \neg C)$ is theorem, but $\square_A(C) \vee \square_A(\neg C)$ is not, it follows that $\square_A (C \vee \neg C)$ does not imply $\square_A(C) \vee \square_A(\neg C)$ for some C. Consequently the second condition for modality holds. That is If A fails to satisfy (**CEM**), then $\square_A$ is a modal operator.

It is evident therefore that if A is either a theorem, or a contradiction, then $\square_A$ will not be a modal operator.

We can also see easily what the truth-value of the counterfactual is, if the antecedent is either a logical truth or a contradiction. The reason is elementary: If the antecedent A is a contradiction then $A \Rightarrow B$, for all B. Therefore, we have $A \square\!\!\rightarrow A \Rightarrow A \square\!\!\rightarrow B$ (by Rule (**D**)). Consequently, since $A \square\!\!\rightarrow A$ is a theorem of VC, so too is $A \square\!\!\rightarrow B$. Therefore $A \rightarrow B \Rightarrow A \square\!\!\rightarrow B$. Moreover, we always have $A \square\!\!\rightarrow B \Rightarrow A \rightarrow B$, so that

A$\Box\!\!\rightarrow$ B and the material conditional $A \rightarrow B$ are equivalent in VC. On the other hand, if the antecedent A is a logical truth, then since A$\Box\!\!\rightarrow$B $\Rightarrow$ $A \rightarrow B$ for all B, and $A \rightarrow B \Rightarrow B$, we have A$\Box\!\!\rightarrow$B $\Rightarrow$ B. We also have the converse: since B is equivalent to $B \wedge A$, which in turn implies A$\Box\!\!\rightarrow$B, by Axiom (6). Thus we have the result that if A is a logical truth, then A$\Box\!\!\rightarrow$B is equivalent (in VC) to the material conditional $A \rightarrow B$ (which in this case is just B). In short then, in the case of impossible or necessary antecedents, the operator fails to be modal, and the counterfactual conditional is equivalent to the material conditional and is evaluated as such, without benefit of creative world making.

As in the intuitionist case we have a field of modal operators, which have a role in explaining some interesting features of counterfactuals, and in defusing certain questions that might seem intractable. For example, it is obvious that although counterfactuals imply their corresponding material conditionals the converse does not hold. Why not? Well if a material conditional implied its corresponding counterfactual, then the counterfactual would be equivalent to the material condition, and as we have seen, the operator associated with the antecedent would not be modal. The modality of this particular operator makes it plausible to attempt a possible world's analysis of them, and it is this modality of this operator that prevents the collapse of the counterfactual into a material conditional.

9. Basic Epistemic Conditional Model, ***ECM***. Finally it is worth noting another significant example of a logical system, this time an epistemic logic, where no modal appears to be present, but attention to the conditional of that logic yields an unusual, unstudied class of modal operators . I have in mind Horacio Arlo Costa's penetrating study of the conditionals involved in Isaac Levi's account of belief revision.[7]

Roughly speaking, let L_0 be a language that contains a complete set of (Boolean)

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[7] "*Belief Revision Conditionals: Basic Iterated Systems*.", (1999), *Annals of Pure and Applied Logic*, 96: 3-28].

connectives together with the constants *verum* $\top$, and *falsum* $\perp$, and let the language $L_>$ be the smallest extension of L_0 by the addition of a conditional connective ">", such that A > B is in it for any A and B in it, and such that it is closed under the Boolean connectives. Then the conditional system *ECM* is the smallest set of sentences of $L_>$ which is closed under the rules of Modus Ponens and (**RCM**), and which contains all the instances of axioms (1) –(3), all classical tautologies, and the substitution instances of all of those tautologies in the language $L_>$. The axioms are: for all A, B, and C,

1. $A > \top$.
2. $(A > B) \wedge (A > C) \rightarrow (A > B \wedge C)$.
3. $\neg (A > C) \leftrightarrow (A > \neg C)$, where C is any sentence in a set of sentences *BC* ,

and the two rules are

Modus Ponens, and

RCM If $(B \rightarrow C)$ is a theorem, then so too is $(A > B) \rightarrow (A > C)$.

The to *BC* in axiom 3 is to a proper sub-language of $L_>$ which extends L_0 this way: It is a conditional language such that if A and B are in L_0, then the conditional A > B is in *BC*, and if C and D are any members of *BC*, then the conditional C > D, the negation $\neg$C, and the conjunction $C \wedge D$ are also in *BC*.

For this epistemic theory, let us form the prefixing operator "H_A", which to every sentence B, assigns the epistemic conditional "A > B"as its value. As in the case of the conditionals of Intuitionism and counterfactuals, we again obtain an infinite number of modal operators, mainly by appeal to (**RCM**) to establish the first condition (M1) of modality, and the failure of (**CEM**) to yield the second condition (M2).

We can now shed some light on the very plausible assumptions of this epistemic logic, if we recast its axioms and rules using the various modal operators H_A.. The first axiom is equivalent to the claim that for any theorem C , and any A, $H_A(C)$ is also a theorem. This is just the requirement of *necessitation* for modal operators. The second axiom says that for any A, B, and, C: $H_A(B) \wedge H_A(C) \Rightarrow H_A(B \wedge C)$, which expresses the *normality* of the modal operators H_A. The third axiom says that for all members C of *BC*, and any A, either $H_A(C)$ or $H_A(\neg C)$. It is an axiom that is taken as typical of epistemic logics, and precludes a type of indeterminacy for conditionals. It says that when testing for whether a conditional is accepted or not, the answer will be one way or the other. There is no suspension of judgment when B is in *BC*. This axiom is different from the requirement of conditional excluded middle (**CEM**), because it only holds for a special subset of all the sentences of $L_>$.

The representation of the epistemic conditional A > B, as the explicitly modal statement $H_A(B)$, gives it an additional interest. We see immediately that the epistemic conditional ofArlo Costa's ***ECM*** cannot be the same as the material conditional, because the operator "If A then" associated with the material conditional is, as we have seen, not a modal operator. In contrast, the operator H_A associated with the conditional of ECM is a modal operator.

We can now express the difference between the conditionals of Arlo Costa's ECM, those of Intuitionism, and of Lewis' counterfactual theory VC. The modal operators associated with the antecedents of those conditionals have different characteristic features. For example the epistemic modal H_A couldn't be identical with the modal φ_A of Intuitionism. As we have seen that according to (***ECM***), $H_A(B) \wedge H_A(\neg B)$ is inconsistent for any element of *BC*. But is easily seen that for the Intuitionistic φ_A, there are members B in BC for which $\varphi_A(B) \wedge \varphi_A(\neg B)$ is not inconsistent.

It is also worth noting that the modal operators $\square_A$ associated with the antecedents of counterfactual conditionals are different from the modals φ_A associated with Intuitionism. The reason is that we know that $B \Rightarrow \varphi_A(B)$ for all A and B. Assuming that the modals have the same character (obey the same laws) then $B \Rightarrow A \square\!\!\rightarrow B$, that is $B \Rightarrow \square_A(B)$ for all A and B. Then, $\neg B \Rightarrow A \square\!\!\rightarrow \neg B$, that is, $\neg B \Rightarrow \square_A(\neg B)$, for all A and

B. Since VC is classical, we conclude that B $\Rightarrow \Box_A(B) \vee \Box_A(\neg B)$ is a theorem for all A and B. But that is impossible – for (CEM) fails in VC.

There is a certain subtlety that is involved in our description of these differences. We compared two modal operators; one on an Intuitionistic structure, and the other on the structure VC which is classical. We have concluded that the two modals are different because B $\Rightarrow \varphi_A(B)$ holds in the Intuitionistic structure, but B $\Rightarrow$ A$\Box\!\!\rightarrow$B fails in the Classical structure VC. We have not assumed that both modals are acting on one and the same structure, and differ on it. This particular comparison is impossible to make, since of course, no implication structure can be both Intuitionistic and classical.