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Structuralist Logic: Implications, Inferences, and Consequences

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Abstract. On a structuralist account of logic, the logical operators, as well as modal operators are defined by the specific ways that they interact with respect to implication. As a consequence, the same logical operator (conjunction, negation etc.) can appear to be very different with a variation in the implication relation of a structure. We illustrate this idea by showing that certain operators that are usually regarded as extra-logical concepts (Tarskian algebraic operations on theories, mereological sum, products and negates of individuals, intuitionistic operations on mathematical problems, epistemic operations on certain belief states) are simply the logical operators that are deployed in different implication structures. That makes certain logical notions more omnipresent than one would think.

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1. The structuralist background

The kind of Structuralist Logic that I have in mind begins with *implication structures*, those ordered pairs consisting of a non-empty set, together with an *implication relation* on it. One of the key ideas of a structuralist theory of logic is that implication is central to logic, and that the logical operators, as well as modal operators generally, can be characterized as such, by the way they interact with respect to implication. Also key is the idea that there may be many different implication relations, and the results for each of the operators can nevertheless look very different, given a difference in the implication relation. Sometimes the operators will be just what we expected them to be, and sometimes, they will not "look"

This is essentially based upon the text of an address given at the first UNILOG Conference, April 2005, in Montreux Switzerland.

like (say conjunction), but will be conjunction nevertheless. I want to explore this particular feature of logical structuralism at greater length: that some theories may contain concepts and operations which are not usually regarded as logical, but are at least as logical in their features as the usual logical operators – they are the logical operators with respect to another implication relation. Throughout the remainder of this essay, we shall say that an **implication structure** $\Im = \langle S, \Rightarrow \rangle$ is any non-empty set S with an implication relation on it, and understand an implication relation to be any relation which satisfies Gentzen's "Structural Conditions": Reflexivity, Projection, Permutation, Simplification, Dilution, and Cut. That is, let S be any non-empty set. We shall say that $\Rightarrow$ is an implication relation on S if and only if for any $A, B, A_1, \ldots, A_n, B_1, \ldots, B_m$, and C in S:[1]

1. Reflexivity. $A \Rightarrow A$, for all A in S.
2. Projection. $A_1, \ldots, A_n \Rightarrow A_k$, for $1 \leq k \leq n$.
3. Simplification. If $A_1, A_1, \ldots, A_n \Rightarrow B$, then $A_1, \ldots, A_n \Rightarrow B$.
4. Permutation. If $A_1, \ldots, A_n \Rightarrow B$, then $A_{f(1)}, \ldots, A_{f(n)} \Rightarrow B$ for any permutation f of $\{1, 2, \ldots, n\}$.
5. Dilution. $A_1, \ldots, A_n \Rightarrow B$ then $A_1, \ldots, A_n, C \Rightarrow B$.
6. Cut. If $A_1, \ldots, A_n \Rightarrow B$, and $B, B_1, \ldots, B_m \Rightarrow C$, then $A_1, \ldots, A_n, B_1, \ldots, B_m \Rightarrow C$

There is a redundancy that can be reduced. *Reflexivity* follows from *Projection*, and *Dilution* follows from *Projection* and *Cut. Projection* follows from *Reflexivity*, *Dilution*, and *Permutation.* Elliott Mendelson[2] has shown that the conditions of *Projection*, *Simplicity*, *Permutation*, *Dilution*, and *Cut* form an independent subset. The six conditions are based upon Gentzen's "structural conditions" some of which go back to his dissertation[3] in which the relation to the work of Paul Hertz is made evident. This notion of implication relation of Gentzen/Hertz is closely related to the concept of the consequence $Cn(A)$ of a set of sentences A, which Tarski introduced at roughly the same time.[4] There is a very natural connection between the Gentzen/Hertz notion of implication and the Tarskian Consequence operation:

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[1]The implication relation is a relation that is meant to be the same as the one indicated by the single arrows used by Gentzen [4] in his sentences (sequenzen) $A_1, \ldots, A_n \Rightarrow B$. It is not to be confused with the material conditional or any other logical connective. This was clearly Gentzen's intention since he held that the logical connectives were each paired with introduction and elimination rules, and he never gave introduction and elimination rules for those arrows.
[2]Private correspondence.
[3]The structural conditions are introduced in Gentzen [5], and the dissertation of Gentzen [4] indicates the reliance on Hertz's idea of expressing dependencies of various kinds of elements by means of special kinds of sentences.
[4]Tarski [17], translated and reprinted in [18]. That essay contains references to the use of the consequence operation even earlier than 1930. Essay V(1935) in [18] also contains references to the final (1929) paper of Hertz [6] on implication, and Essay XVII (1937) in [17] refers to Gentzen [5].

(**GHT**) For any sentences $A_1, A_2, \ldots, A_n$ and B, we have $A_1, A_2, \ldots, A_n \Rightarrow B$ if and only if $B \in Cn(\{A_1, A_2, \ldots, A_n\})$.[5]

The connection is sound if we restrict the As and B to sentences of some formal language. Nevertheless there may be some advantage in the use of the Gentzen/Hertz implication relation for the purposes of a structural account of logic. Both the Gentzen/Hertz implication relation and the Tarskian consequence operation are abstract or general in the sense that they admit of many different kinds of examples or instances. Tarski's work is full of examples of the power of the consequence operator in varied logical and mathematical contexts. But varied as they are, the consequence operators acts on sets of sentences that belong to languages with rules of inference. They are sentences that belong to various specific cases of what Tarski termed "formalized disciplines." In what looks like a contrast, the intended examples of the Gentzen/Hertz implication relations on the other hand include not only the cases mentioned, but some unusual examples as well, which do not seem easily accommodated as cases where one sentence is a consequence of some set of sentences. Here is how Gentzen describes the rather wide scope of the examples covered. He has already described his sentences as having the form $A_1, A_2, \ldots, A_n \rightarrow B$, with the A_i and B as its elements. Of the elements he says

> "We might think of them as events, and the 'sentence' then reads: The happening of the events $A1, \ldots,$ and A_n causes the happening of B.
>
> The 'sentence' may also be understood thus: A domain of elements containing the elements $A_1, \ldots, A_n$ also contains the element B.
>
> The elements may furthermore be thought of as properties and the 'sentence' can then be interpreted thus: An object with the properties $A_1, \ldots, A_n$ also has the property B.
>
> Or we imagine the elements to stand for 'propositions,' in the sense of the propositional calculus, and the 'sentence' then reads: If the propositions $A_1, \ldots, A_n$ are true, then the proposition B is also true.
>
> Our considerations do not depend on any particular kind of informal interpretation of the 'sentences', since we are concerned only with their formal structure"[6].

Thus what the sentences may relate, their elements, could be events, they could be any kind of individual or object that can be a member of a set (anything belonging to a "domain"), they could be properties, or anything that might be

[5] J.-Y. Beziau [2] in his insightful paper, takes the story from the Gentzen/Heyting implication and Tarski consequence relation through the intricate developments to Algebraic, Abstract, and Universal Logic. D. Scott [15] has an interesting comparison of the two concepts, and in passing, generalizes the Tarskian consequence operation into a version of a Gentzen symmetric sequent.

[6] Gentzen [4].

considered the "propositions" of the propositional calculus. For the purposes of a structuralist account of logic, the elements of Gentzen's sentences (at least in his dissertation) appear to offer an edge in generality over the use of consequence operations. In that account, as we shall see, we do want to explore the possibility of implication relations for events, objects, sets, as well as the elements of formalized disciplines. There is a second consideration why the Gentzen/Hertz implication relations may be more suitable for the exposition of structuralist logic. One of the aims is to define the various logical operators in terms of the implication relation on various implication structures. It would be hopelessly circular if we tried to define the various logical operators using consequence operators on sets of sentences of formal disciplines which already used those operators for their expression.[7]

A key part of a structuralist approach is to define the various logical operators as special functions defined on implication structures. It will be helpful in what follows to use *Conjunction*, *Negation*, and *Disjunction* as examples of the way these characterizations work. Universal and existential quantification are part of the structuralist story but will not be defined or discussed here.[8]

CONJUNCTION: The conjunction operator on an implication structure $\Im$ is a function of two arguments, such that for any A and B in S, $C(A, B)$ is a conjunction of them if and only if the following two conditions are satisfied:

1. $C(A, B)$ implies A as well as B, and
2. If it exists, it is the weakest member of the structure to satisfy this condition.

That is, if T is any member of the structure, and if T implies A, as well as B, then T implies $C(A, B)$.

NEGATION: The negation operator on a structure is a function of a single argument such that for any member A of the structure, it's negation $N(A)$ (if it exists) is a member of the structure that satisfies two conditions:

1. A together with it's negation imply all the members of the structure, and
2. For any T in the structure, if T together with A implies everything in the structure, then T implies $N(A)$. The negation of A is the weakest member of the structure to satisfy the first condition.

DISJUNCTION: The disjunction operator on an implication structure is a function of two arguments, such that for any A and B in S, $D(A, B)$ is a disjunction of them if and only if the following two conditions are satisfied:

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[7] I suppose that it might be possible to understand (GHT) in such a way that the $A_{i's}$ and B can be as varied as the Gentzen/Hertz relations allow, and still make sense for the consequence operators. Something would have to be done about relaxing the requirement on consequence operations so that they are not restricted to subsets of a set of sentences S, which has some rule of inference by which some sentences of S can be obtained from others. If that could be done, then either one of these concepts would of course be a suitable framework for developing a structuralist account of logic.

[8] A fuller account can be found in Koslow [11], Chapter 3.

1. For any T in the structure, if A implies T and B implies T, then their disjunction (if it exists) implies T, and
2. It is the weakest member of the structure to satisfy the first condition.

The logical operators in this theory are all defined in an interesting uniform way (whose algebraic description and connection with Gentzen's Introduction and Elimination Rules cannot be explained here)[9]. It is easily seen that if there are several members of a structure that meet the conditions of being (say) the negation of some element, then they have to be equivalent.

It is the possibility of variations either in the sets or variations in the implication relations of implication structures that gives rise to a number of interesting ways of looking at familiar theories and concepts that at first sight do not seem to have any logical content, but turn out to be matters of logic after all. Before we look at some examples, perhaps it is helpful to indicate some of the more general results for implication structures that are relevant to a discussion of specific examples:

1. **Scope.** The usual examples of the sentential calculus (classical and intuitionistic) and first-order quantification can be represented as implication structures, and it seems, substructural logics can be represented as well.
2. **Truth-values.** Despite the fact that implication relations can hold between items that are not truth bearers, a notion of truth-value assignments can be given in structuralist terms, for which the following completeness theorem can be proved: For any implication structure $\Im = \langle S, \Rightarrow \rangle$, a truth-value assignment τ is given by a partition of S into two non-empty exclusive and exhaustive sets L and K (where the $L's$ are closed under the implication relation $\Rightarrow$), and any element A of the structure is true under the assignment τ, if A is in L ; false otherwise. Then we have the following result:

   (Lindenbaum–Scott Completeness): $A_1, \ldots, A_n \Rightarrow B$ if and only if for every truth-value assignment τ, if all the $A_{i's}$ are true under the assignment τ , then so too is B.

3. **Completeness and the intuitionistic connection.** Roughly speaking despite the variety of elements that are in various implication structures, it can be proved that if we limit ourselves to those structures in which the logical operators always have values, then those theses which hold in *all* implication structures are precisely those which correspond to theses of Intuitionistic Logic, and those theses which hold in all classical implication structures (those in which the double negation of any A implies A) are precisely those which correspond to theses of Classical logic.

[9]See Koslow [11].

4. **Duality.** It can be shown that to every implication relation on a structure, there corresponds its dual, which is also an implication relation. More exactly, let $\Im = \langle S, \Rightarrow \rangle$ be an implication structure on a set S, with the implication relation $\Rightarrow$. Let $\Rightarrow^{\wedge}$ be the dual of the implication relation $\Rightarrow$, defined as follows:

   For any $A_1, A_2, \ldots, A_n$ and B in S: $A_1, A_2, \ldots, A_n \Rightarrow^{\wedge} B$ if and only if for every T in S, if all the $A_i \Rightarrow T$, then $B \Rightarrow T$.[10]

   It easily follows that for the case of single antecedents, for any A and B, $A \Rightarrow B$ if and only if $B \Rightarrow^{\wedge} A$, and if the disjunction of $A_1, A_2, \ldots, A_n$ exists in the structure, then $A_1, A_2, \ldots, A_n \Rightarrow^{\wedge} B$ if and only if $B \Rightarrow A_1 \vee A_2 \vee \ldots \vee A_n$.

   Given the various definitions of the logical operators on any implication structure, we can obtain the duals of those operators by replacing the implication relation $\Rightarrow$ by its dual $\Rightarrow^{\wedge}$. It is easily seen that by using our definitions of conjunction and disjunction, that the dual of conjunction is disjunction and conversely. That is not news! Much more interesting is the case when the implication structure is not classical. Here N^* the dual of the negation operator N, is different from it, and the two operators seem to split classical negation between them: the negation operator in any structure (classical or not) always yields that any A implies its double negation. In the non-classical case however, A implies $NN(A)$, for all A, but not conversely. While the dual picks up the classical slack: It's always the case that (1) $N^*N^*(A)$ implies A, and (2) A or $N^*(A)$ is a thesis, and (3) $N(A)$ implies $N^*(A)$.[11] It is easily seen that in any implication structure, if A implies B, then $N^*(B)$ implies $N^*(A)$. So we see that on any implication structure, there are some fairly fundamental properties which negation and its dual share, and there are some other central properties of negation which get divided up between negation and it's dual. Perhaps the nicest simple result about the dual of non-classical negation is this exact characterization of the classical implication structures:

   An implication structure is classical if and only its negation operator is identical with its dual.

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[10]The *locus classicus* for the notion of duality of the consequence relation is Wojcicki [19]. The version given here, for implication relations, is in Koslow [11], pp. 60–66, and [12], p. 117. It is equivalent to the idea that in any implication structure $\langle S, \Rightarrow \rangle$, the dual of $\Rightarrow$ is that implication relation on S that extends all the implication relations $\Rightarrow^*$ such that $A \Rightarrow^* B$ if and only if $B \Rightarrow A$, where of any implication relations $\Rightarrow'$ and $\Rightarrow$ on S, the implication relation $\Rightarrow^*$ extends the implication relation $\Rightarrow$, if and only if for any $A_1, A_2, \ldots, A_n$ and B in S, if $A_1, \ldots, A_n \Rightarrow B$, then $A_1, \ldots, A_n \Rightarrow' B$.

[11]See Lopez-Escobar [13] on the important question of whether the dual of intuitionistic negation is intuitionistically acceptable in quantified intuitionistic logic.

It's also worth noting that there is a paraconsistent connection: in any non-classical implication structure the dual negation is non-explosive: A together with $N^*(A)$ does not imply every member of the structure. If we widened the notion of logical operators on a structure so that it included not only those operators with respect to the implication relation ($\Rightarrow$) of the structure, but all the duals of those operators, then we would have in any non-classical structure both the possibility of two negation operators on the structure: N and its dual N^*. This would allow that non-classical implication structures are paraconsistent in that they have a non-explosive negation operator on them (if the dual negation exists). Of course they also have an explosive negation on them. I'm not sure that this kind of ecumenical possibility would be acceptable to most or any paraconsistent logicians. The objection would be that the explosive operator, no matter what you call it, is not paraconsistently acceptable. Of course grounds would have to be given in support of such a rejection. The situation might be paralleled in a way by some Intuitionist reaction to such a two-negation structure. That objection might be that the dual of the non-classical negation operator may be an operator, but it is not an intuitionistically acceptable operator. Of course the case would need argument. If you took the non-classical structure with the two negations on it, and considered the structure with just the dual of the non-classical negation operator on it, then that structure with just N^*, the dual of the negation operator on it, would be paraconsistent. Of course it would not be intuitionistically acceptable since N^* is not explosive. Nevertheless this construction would show how to get some paraconsistent structures by modifying non-classical ones, and intuitionistic ones in particular.

2. Familiar logic in different settings

In what follows I will consider three examples where something extralogical seems to be the object of study, but structuralism indicates otherwise. In a sense there's more logic going on in these examples than meets the eye.

2.1. Gärdenfors and epistemic implication

Peter Gärdenfors[12] considered (1) a set of belief states K of an agent and (2) a notion of proposition that is closely tied to those belief states in order to define an epistemic consequence relation among the propositions. If S is a set of belief states, then the set of Gärdenfors propositions G, have as members those functions that map S to S which are idempotent and commute with each other. We let fg be the functional composition of any two functions f and g of G, so that for any belief state K, fg is a function mapping S to S such that $fg(K) = f(g(K))$. As usual, idempotency requires that $ff = f$, and commutativity requires that $fg = gf$.

[12] P. Gärdenfors [3], pp. 1–10.

It is readily apparent that there is something that seems quite arbitrary and unmotivated about these two conditions for propositions. However the situation can be strongly motivated from a structuralist perspective. On the set of propositions, Gärdenfors defined an epistemic consequence relation between his propositions this way: $f \Rightarrow g$ if and only if in every belief state K, if f is accepted as known in K, then so too is g.

Clearly this relation is epistemic, since it is equivalent to a familiar weak epistemic closure condition: If f implies g, and f is known, then so too is g. Gärdenfors also explained what he meant by knowledge of a proposition:

> A proposition f is accepted as known in a belief state K if and only if $f(K) = K$.

That is, the proposition f is known in a belief state K if and only if the belief state is a fixed point of the proposition (function) f. It follows immediately from his definition, and this explanation, that the Gärdenfors consequence relation is equivalent to a functional equation between certain propositions:

> $(G) f \Rightarrow g$ if and only if $gf = f$.

I would like to determine what the logical operators look like in this kind of setting. In order to do that we have to generalize Gärdenfors' Epistemic Consequence relation. As it stands it is not an implication relation, since they include cases of multiple premises, and Gärdenfors' proposal covers only the single premise case. This can easily be remedied by generalizing the Gärdenfors' consequence relation to a multiple premise version acceptable to Gärdenfors[13]:

> $(GG) f, g, \ldots, h \Rightarrow k$ if and only if $kfg \ldots h = fg \ldots h$, for any $f, g, \ldots, h$ and k in G.

We shall say that an ordered pair of the set of propositions, together with the generalized relation GG is a *Gärdenfors Implication Structure* provided that the generalized relation is an implication relation. Before we turn to that issue we should note that the generalized relation implies a strong form of epistemic closure:

> If in state K each of the propositions f, and g, and, $\ldots$, and h are known, and $f, g, \ldots, h$ imply k, then the proposition k is also known in state K.

[13] In private correspondence.

We hasten to add that we are not endorsing the principle of generalized epistemic closure. We are not denying it either. It is a substantive claim and it is worth noting that it does follow from the generalized Gärdenfors implication relation. To the extent that epistemic closure is a problem, so too is the Gärdenfors implication relation.

Gärdenfors has claimed that his set of propositions are those functions that are idempotent and commute with each other, and that the conjunction of propositions f and g is just their functional composition. This looks like a compounding of implausible proposals. Nevertheless it is a theory that has a remarkable coherence. And to explain why that is so, we need to note two things: If the Gärdenfors propositions are idempotent and commute with each other, then it follows that (1) the generalized relation (GG) is indeed an implication relation in our sense of that term. It is a simple matter to verify that the six conditions of *Reflexivity*, *Projection*, *Dilution*, *Simplicity*, *Permutation*, and *Cut* all hold. It also follows that (2) the conjunction operator on the Gärdenfors implication structure is functional composition.

To see why this is so, we look to our characterization of the conjunction operator on implication structures. The first condition requires that the conjunction $C(f, g)$ of the members f and g of the structure implies f and implies g. The functional composition fg certainly implies f because that holds if and only if ffg is equal to fg. That however holds by idempotency. It is also true that fg implies g, because that holds if and only if gfg equals fg. But that holds by comutativity and idempotency. So the first condition of conjunctions holds. The second condition requires that if any member h of the structure implies f as well as g, then h implies the functional composition fg – that is, of all the members of the Gärdenfors implication structure to imply both the propositions (functions) f and g, their functional composition fg is the weakest.

The reason is fairly direct: Notice that if h implies f, then $fh = h$, and if h implies g, then $gh = h$. Therefore $fgh = fh = h$. Consequently, h implies fg. Moreover the conclusion can be strengthened: we can prove from the general definitions of the logical operators on structures, that the values of them are unique up to the equivalence on the structure. In this case the conjunction $C(f, g)$ of two propositions is identical to their functional composition. The proof is again obvious: suppose that f implies g and g implies f. Then $gf = f$ and $fg = g$. Therefore (using the comutativity of the functions), $f = g$. Thus, the functional composition fg is the conjunction $C(f, g)$ of the two propositions.

What the structuralist account show us is that once we use the features of idempotency and commutativity to secure a special implication relation, then those features are not only mathematical requirements on functions, they are also the features of the operator of logical conjunction, only in this special setting.

2.2. Tarski's calculus of systems

The seminal idea of a (Tarskian) theory or system is well known. If S is a nonempty set of sentences, then a (deductive) consequence operator is a mapping of the set of subsets of S to itself, such that the following conditions are satisfied where X and Y are any subsets of S:

1. $X \subseteq Cn(X)$.
2. If $X \subseteq Y$ then $Cn(X) \subseteq Cn(Y)$.
3. $CnCn(X) \subseteq Cn(X)$.

It should be noted that the set of consequences of the set theoretical union of X and Y, $Cn(X \cup Y)$ in general is not the union of the set of consequences of X with the set of consequences of Y. According to the first condition, every subset X of S is a subset of $Cn(X)$, its set of consequences, also called the closure of X. If X is equal to its own closure, it is called closed. *Tarskian Theories* or *Systems* are those subsets of S which are closed.

Tarski's Calculus of Systems or Theories is the systematic study of the theories of S with the use of certain mathematical operations that map theories to theories. With the exception of set theoretical intersection, the operations which Tarski introduced are not the usual Boolean ones of union and relative complement $(S - X)$. In the case of intersection, the intersection of two theories is a theory. However the union of two theories in general may not be a theory (there may be consequences of the union of theories X and Y that are not consequences of X nor of Y). And the relative complement of a theory will not in general be a theory. It is the genius of Tarski to have introduced the operators "the logical sum", and the "logical complement" which yield theories when operating on theories. Together with intersection they provide the theories of S with an algebraic structure of great importance in logic.

By the logical sum of the theories X and Y, Tarski meant the closure of the union of the two – that is, $X \oplus Y = Cn(X \cup Y)$. The logical complement is slightly more difficult to express. The logical complement of any theory X is the closure of the union of all those theories whose intersection with X is the set L of all theses of the set S (those elements of S that are implied by everything in S).

So it seems for all the world that these various operations are an addition to the logical connectives. And the Calculus of Systems looks like an application of first order logic with the "usual" logical operators on the sentences, and the algebraic, apparently extra-logical operations on theories. All that is of course true, but there is something deeper that has taken place: the various operations on theories are just the logical operators in a different setting.

To see this, consider a Tarski Implication Structure consisting of the set T of theories of some set S, together with an implication relation "$\Rightarrow$" defined on theories this way:

> For any theories $U, V, W, X, Y, Z, \ldots,$ we let $U, V, \ldots, W \Rightarrow Z$ if and only if $U \cap V \cap \cdots \cap W \Rightarrow Z$, and $U \Rightarrow Z$ if and only if $U \subseteq Z$.

One theory implies another if and only if it is a subset of the other, and a finite number of theories imply Z if and only if their intersection is a subset of Z.

In an implication structure of Tarskian theories, the logical operators of conjunction, disjunction, and negation as characterized for any structure, turn out, on this particular Tarskian structure to be exactly the algebraic operations of intersection, logical sum, and the logical complement of theories respectively. From a structuralist perspective, Tarski's instincts were unerring. That is, not only are the logical operators equivalent to the Tarskian ones, they can be proved to be identical to them, (we omit the proofs, which are rather straightforward):

1. $C(U, V) = U \cap V$.
2. $D(U, V) = Cn(U \cup V) = U \oplus V$, and
3. $N(U) = Comp(U)$.

If we hadn't the genius of Tarski to guide us, we might have nevertheless come to rediscover, in hindsight, some of the important mathematical operations on theories that he introduced.

2.3. Mereology

Let us turn now to a rather simplified form of the *Calculus of Individuals* introduced by Lesniewski and developed by H. Leonard, Nelson Goodman, and a host of later writers. Though simplified, it suffices for showing that Mereology is not just a first order theory supplemented with special operators that map individuals to individuals. The central concepts of Mereology – the *Product*, *Sum*, and *Negate* of individuals are in fact just the familiar logical operators in a different setting where they act on individuals.

Let us define a Mereological Implication Structure to consist of a non-empty set of individuals S with a special mereological implication relation on it that is defined with the help of a whole-part relation. A whole-part relation on S will be any binary relation "A is a part of B", satisfying four conditions: (1) Reflexivity, (2) Transitivity, (3) Identity: A is identical with B if and only if each is a part of the other, and (4) If every part of A has a part which is also a part of B then A is a part of B. A mereological implication relation on S is defined this way:

1. Any individual A implies an individual B if and only if it is a part of it. And

2. A finite group of individuals implies an individual B if and only if every individual that is a part of every member of that group is also a part of B.

We now need to define some additional mereological notions: two relations (*overlap* and *discreteness*) and three operators (Sum, Product, and Negate): For the relations we shall say that A *overlaps* B ($A \circ B$) if and only if they have some part in common, and that A and B are discrete ($A \sharp B$) if and only if they have no part in common. For the operations, we shall say that

- The sum of A and B – **SUM(A,B)** – is that individual such that an individual overlaps it if and only if it either overlaps A or it overlaps B.
- The product of A and B – **PRO(A,B)** – is that individual whose parts are exactly those that are the parts of both A and B.
- The negate of A – **NEG(A)** – is that individual whose parts are exactly those that are discrete from A.

It is easily seen that the mereological operations, **SUM**, **PRO**, and **NEG** on a mereological implication structure, are identical, and not just equivalent to the logical operators of disjunction, conjunction, and negation respectively, It is worth mentioning that Tarski had already noted that the negation (the logical complement) of theories is non-classical. In constrast, the negation operator in a mereological implication structure is classical; the negate of the negate of any individual A is identical to A.

3. Historical intuitionism

We have seen how the structuralist account of the logical operators allows for the possibility that there are many different kinds of things that can be negations, conjunctions, disjunctions or conditionals. If one insisted that the logical operators are limited to sentences, statements or propositions (having truth values), then the examples we considered would not be structural insights, they would be at most be merely formal illusions. Such a doctrine would bar anything but statements or sentences from having the logical complexity that logical operators provide. As a consequence of such a doctrine we would also have to discount an important part of the history of logic. We would have to discount much of what was critical for historical intuitionism, for it would not count as logic. I have in mind Kolmogorov's use of problems (which can be solved or not, where the demonstration that there is no solution, is a solution), Becker's use of Husserlian expectations (which may be satisfied or fulfilled, or not), and Heyting's use of propositions in his exposition of the fundamental concepts of intuitionistic logic. Today we have time only to do little more than mention Kolmogorov's formulation of his *Calculus of problems*. As he said,

> "In addition to theoretical logic, which systematizes the proof schemata of theoretical truths one can systematize the schemata of the solution of problems ..."[14]

[14]Kolmogorov [9]; translation from the German in P. Mancosu [14], pp. 328–334.

After noting that the Calculus of Problems is formally identical to Heyting's formulation of Intuitionistic logic, he proceeded to argue for the stronger point:

> "[I]ntuitionistic logic should be replaced by the calculus of problems, for its objects are in reality not theoretical propositions but rather problems"[15]

What is explicit and clear from Kolmorogov's presentation is that in this case it is not propositions which figure in the logical operators, but mathematical problems, and the logical operators become problem-forming operators. There is a wistful footnote in which Kolmogorov says that Heyting developed a view which was closely connected to the problems interpretation, but that Heyting still had not yet made a clear distinction between problems and propositions.[16] Becker's[17] view on the other hand was that intuitionistic logic concerned Husserlian intentions. Heyting, at about this time, wrote about intuitionistic propositions in a way which managed to group together the use of problems, the use of intentions, and his own idea of propositions as if they amounted to the same thing, and were the proper topic of intuitionistic logic. Thus he wrote that

> "A proposition p like, for example, 'Euler's constant is rational' expresses a problem, or better yet, a certain expectation (that of finding two integers a and b such that $C = a/b$) which can be fulfilled [réalisée] or disappointed [déçue]."[18]

Even if Heyting later changed his mind, still he did think that Becker and Kolmgorov's views were possible ways of understanding how the logical operators were being used or could be used or even should be used in Intuitionist logic. All of these possibilities however are ruled out by a doctrine that insists that the logical operators are restricted to sentences, statements, or propositions (which are assumed to have truth values). The structuralist account we have described doesn't require us to be that dogmatic about logical matters. There are many implicational structures. Intuitionistic logic is an important one, and it's historical versions are as well. What we need is an account that, for doctrinal reasons, does not eject these examples from our logical tradition. The structural approach provides one.

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[15] Mancusu [14], p. 328.
[16] Kolmogorov [9]; translation from the German in P. Mancosu [14], pp. 328–334.
[17] The text is in O. Becker [1] and a particularly trenchant passage on this point is translated in Mancosu [14], p. 279.
[18] Heyting [7]. Translated from the French by Amy L. Rocha, and published in Mancosu [14], p. 307.

References

[1] O. Becker, *Mathematische Existenz*, Jahrbuch für Philosophie und phänomenogische Forschung, 8:439–809 (1927).

[2] J.-Y. Beziau, *From Consequence Operator to Universal Logic: A Survey of General Abstract Logic*, Logica Universalis, J.-Y. Beziau (Ed.), Birkhäuser Verlag, Basel, 2005.

[3] P. Gärdenfors, *Dynamics of Belief as a Basis for Logic*, British Journal for the Philosophy of Science, 35:1–10 (1984).

[4] G. Gentzen, *Über die Existenz unabhängiger Axiomsysteme zu unendlichen Satzsystemen*, Mathematische Annalen, 107:329–350 (1933); reprinted and translated (1969) as *On the Existence of Independent Axiom Systems for Infinite Sentence systems.* In The Collected Papers of Gerhard Gentzen, M. E. Szabo (Ed.) pp. 29–52. Amsterdam, North Holland.

[5] G. Gentzen, *Untersuchungen über dass logische Schliessen*, Mathematische Zeitschrift, 39:176–210, 405–31 (1934), translated and reprinted as *Investigations into Logical Deduction*, in M. E. Szabo (1969).

[6] P. Hertz, *Über Axiomensystemen für beliebige Satzsysteme*, Mathematische Annalen, 101:457–514 (1929).

[7] A. Heyting, *Sur la logique intuitionniste*, Académie Royale de Belgique, Bulletin, 16:957–63 (1930).

[8] A. Heyting, *Die intuitionistische Grundlegung der Mathematik*, in Erkenntnis, 2:106–115 (1931).

[9] A. Kolmogorov, *Zur Deutung der intuistischen Logik*, Matematische Zeitschrift, 35:58–65 (1932).

[10] A. Kologorov, *On the Interpretation of Intuitionistic Logic*, translated from the German in P. Mancosu in [14].

[11] A. Koslow, *A Structuralist Theory of Logic*, Cambridge University Press, New York, 1992.

[12] A. Koslow, *The Implicational Nature of Logic: A Structuralist Account*, in European Review of Philosophy, The Nature of Logic (A. C. Varzi, Ed.), CSLI Publications, Stanford, Volume 4, 1999, pp. 111–155.

[13] E. G. K. Lopez-Escobar, *Intuitionistic Connectives* (1981) preprint and published as *On Sentential Intuitionistic Connectives.* I in Columbian Magazine of Mathematics, 19:1–2 (1985).

[14] P. Mancosu, *From Brouwer to Hilbert, The Debate on the Foundations of Mathematics in the 1920s*, Oxford University Press, New York, 1998.

[15] D. Scott, *Completeness and Axiomatizability in Many-Valued Logic* in Proceedings of the Tarski Symposium, L. Henkin et al. Eds., Providence, American Mathematical Society (1974), pp. 411–435.

[16] M. E. Szabo, *The Collected Papers of Gerhard Gentzen*, North Holland, Amsterdam, 1969.

[17] A. Tarski, *On Some Fundamental Concepts of Metamathematics*, Translated from *Über einige fundamentale Begriffe der Metamathematik*, Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie, 23(iii):22–29 (1930). Translation in [18], Essay III.

[18] A. Tarski, *Logic, Semantics, Metamathematics*, Translated by J. H. Woodger, Second Edition, Edited and Introduced by J. Corcoran, Hackett Publishing Company, Indianopolis, Indiana, 1983.

[19] R. Wojcicki, *Dual Counterparts of Consequence Operations*, Bulletin of the Section of Logic (Polish Academy of Sciences) 2:54–57 (1973).

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