Abstract

"A dynamic analysis of human welfare in a warming planet"

Humberto Llavador (Universitat Pompeu Fabra)

John E. Roemer (Yale University)

Joaquim Silvestre (University of California, Davis)

Climate science indicates that climate stabilization requires low GHG emissions. Is this consistent with nondecreasing human welfare?

Our welfare index, called *quality of life* (QuoL), emphasizes education, knowledge, and the environment. We construct and calibrate a multigenerational model with intertemporal links provided by education, physical capital, knowledge and the environment.

We reject discounted utilitarianism and adopt, first, the *Intergenerational Maximin* criterion, and, second, *Sustainable Growth Optimization*, that maximizes the QuoL of the first generation subject to a given future rate of growth. We apply these criteria to our calibrated model via a novel algorithm inspired by the turnpike property.

The computed paths yield levels of QuoL higher than the year 2000 level for all generations. They require the doubling of the fraction of labor resources devoted to the creation of knowledge relative to the reference level, whereas the fractions of labor allocated to consumption and leisure are similar to the reference ones. On the other hand, higher growth rates require substantial increases in the fraction of labor devoted to education, together with moderate increases in the fractions of labor devoted to knowledge and the investment in physical capital.

Keywords: Quality of life, climate change, education, maximin, growth.

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"A DYNAMIC ANALYSIS OF HUMAN WELFARE IN A WARMING PLANET" [1]

Humberto Llavador (Universitat Pompeu Fabra)
John E. Roemer (Yale University)
Joaquim Silvestre (University of California, Davis)

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1. Introduction

Human activity at any moment of time influences future possibilities for welfare through the creation and the destruction of various forms of capital. Some of the benefits of an investment accrue within the lifetime of the generation that makes it. Others are intergenerational: the benefits of accumulated knowledge and of preserved natural environments extend far into the future, and therefore have the character of intergenerational public goods. For instance, the knowledge acquired at a given moment can be used with little additional effort at future dates. And many forms of physical capital, such as infrastructure, are useful beyond the lifetimes of the generation that produces them.

But, on the negative side, current production and consumption also deplete nonrenewable resources and deteriorate the environment. The extinction of, say, an animal species affects both the current generation and all subsequent ones, constituting an intergenerational public bad. As noted in Nicholas Stern (2007), hereafter referred to as the Stern Review, climate change due to anthropogenic greenhouse gas (GHG) emissions is, currently, "the most important externality." We adopt an inclusive approach that simultaneously considers the major intergenerational public goods and bads.

We propose (Section 2.1 below) an encompassing notion of human welfare that includes both consumption and the quality of the environment as arguments. Less conventionally, we also assume that improvements in knowledge and culture, and in education, directly enrich human quality of life.

Next, we view society as comprised of an infinite sequence of generations, and define (Section 2.2 below) two criteria of intergenerational welfare. The first one is John Rawls's (1971) *maximin* criterion, but applied to the various generations. The maximization of this social welfare function leads to stationary levels of human welfare, thus capturing the notion of *human sustainability*.

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Because the growth in mankind's quality of life is often considered a worthwhile objective, we also consider a second social welfare function that takes as given a predetermined, constant rate of growth. In any event, we avoid the discounted utilitarian approach, which we find unjustified at least for the discount rates commonly used.

We construct (Section 2) and calibrate a dynamic model involving economic and environmental variables. Given the complexity and uncertainties of current climate models, we have adopted a modest goal: We eschew the specification of a physical model of emission-stock interactions, and consider instead a particular path for the environmental variables, which entails very low emissions after 2050, and realistically appears to be feasible given present knowledge of climate dynamics, as represented by the IPCC AR 4 (2007, Working Group I, *The Physical Science Basis*, Chapter 10), hereafter referred to as Gerald Meehl *et al*. (2007).

The economic variables are then endogenous in our optimization programs, while the environmental variables are postulated to follow the prescribed path. We develop a computational algorithm based on the turnpike property (Section 3).

We show that positive rates of growth in human welfare are possible while the first generation experiences a quality of life higher than the reference level. The computed paths involve investments in knowledge at noticeably higher levels than in the past: The fraction of labor resources devoted to the creation of knowledge must be doubled, whereas the fractions of labor allocated to consumption and leisure are similar to those of the reference level. On the other hand, higher growth rates require substantial increases in the fraction of labor devoted to education, together with moderate increases in the fractions of labor devoted to knowledge and the investment in physical capital. We test for robustness of the model calibration and find qualitatively similar results.

Lastly, we comment on the introduction of uncertainty in dynamic welfare analysis (Section 5), and on the relation of our analysis to the Stern Review and the work of William Nordhaus and his collaborators (Section 6), and summarize our results (Section 7).

2. Our approach

2.1. The Quality-of-Life function

A large segment of the literature (e. g., Nordhaus, 2008a) postulates an individual or generational utility function with the consumption of a single, produced good as its only argument (sometimes augmented by leisure time): Improvements in knowledge, education, and the environment are then important only in so far as they make possible the production of consumption goods with less labor time or capital.

In fact, both the consumption of goods and the availability of natural capital positively affect human welfare. Indeed, the spectacular increase of consumption in developed economies during the last century has undoubtedly provided a major welfare improvement (D. G. Johnson, 2000). But, in our view, two other factors have also had major impacts. First are the improvements in life expectancy, health status and infant survival, partly due to the rise in consumption, but to a large extent due to medical discoveries, and their implementation by the public health system.[2] Second is the improvement in literacy and, more generally, in the amount of education received by the average person, which has enhanced not only the productivity of labor but also the quality of life: the contribution of leisure to the quality of life increases as leisure time embodies higher levels of human capital, see Salvador Ortigueira (1999) and Martin Wolf (2007), as well as J. J. Heckman (1976) and Robert T. Michael (1972).[3] In Wolf's words:

> "The ends people desire are, instead, what makes the means they employ valuable. Ends should always come above the means people use. The question in education is whether it, too, can be an end in itself and not merely a means to some other end – a better job, a more attractive mate or even, that holiest of contemporary grails, a more productive economy. The answer has to be yes. The search for understanding is as much a defining characteristic of humanity as is the search for beauty. It is, indeed, far more of a defining characteristic than the search for food or for a mate. Anybody who denies its intrinsic value also denies what makes us most fully human."

Accordingly, we choose to call our human welfare index, to be denoted by the Greek letter Λ, the *quality of life* (QuoL), rather than "utility." Our approach follows the spirit of the Human Development Index (HDI) produced by the United Nations Development Program, which considers three dimensions, namely (a) life expectancy, (b) education, and (c) consumption (GDP per capita). On the other hand, as we discuss in Section 2.2 below, the welfare or the consumption of a generation's children is <u>not</u> an argument in the QuoL function.

The first argument in the QuoL function is consumption. But we emphasize other factors as well:

(i) Education, which modifies the value of leisure time to the individual;

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[2] Jim Oeppen and James Vaupel (2002, p. 1029) report that "female life expectancy in the record-holding country has risen for 160 years at a steady pace of almost 3 months per year."

[3] Increases in the human capital of the parents can also improve the quality of their child-rearing services, a component of the parents' "leisure."

(ii) Knowledge, in the form of society's stock of culture and science, which directly increases the value of life (in addition to any indirect effects through productivity), via improvements in health and life expectancy, and because an understanding of how the world works and an appreciation of culture are intrinsic to human well-being,

(iii) An undegraded biosphere, which is valuable to humans for its direct impact on physical and mental health.[4]

Hence, consumption, educated leisure, the stock of human knowledge, and the quality of the biosphere are arguments in the quality-of-life function. The first two arguments are private goods, and the last two are public goods.

We abstract from all conflicts except for the intergenerational one and, accordingly, we assume a representative agent in each generation. We assume that a generation lives for 25 years, and we formally postulate the following QuoL function of Generation t, $t \geq 1$:

$$\tilde{\Lambda}(c_t, x_t^l, S_t^n, S_t^m) \equiv (c_t)^{\alpha_c} (x_t^l)^{\alpha_l} (S_t^n)^{\alpha_n} (\hat{S}^m - S_t^m)^{\alpha_m},$$

where the exponents are positive and normalized such that $\alpha_c + \alpha_l + \alpha_m + \alpha_n = 1$, and where:

c_t = annual average consumption per capita by Generation t ;

x_t^l = annual average leisure per capita, in efficiency units, by Generation t;

S_t^n = stock of knowledge per capita, which enters Generation t's quality-of-life function and production function, understood as located in the last year of life of Generation t,

S_t^m = total CO_2 in the atmosphere above the equilibrium pre-industrial level, in GtC, which is understood as located in the last year of life of Generation t;[5] and

$\hat{S}^m$ = "catastrophic" level of CO_2 in the atmosphere above the pre-industrial level.

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[4] This is captured in the Cost-Benefit literature on global warming by the computation of the so-called "noneconomic effects."

[5] The preindustrial values for the CO_2 stock are taken to be 595.5 GtC or 280 ppm. To convert our S_t^m into CO_2 ppm, add 595.5 to S_t^m and multiply by 0.47. To convert a number of CO_2 ppm into our S_t^m, subtract 280 from it and multiply by 2.13. The presence of the stock of CO_2 in the QuoL function captures our view that environmental deterioration is a public bad in consumption (as well as in production), contrary to the modeling of Nordhaus (1994, 2008a) and Nordhaus and Joseph Boyer (2000), where it is only a public bad in production.

2.2. Optimization programs: **Sustainable quality of life and sustainable growth**

We are concerned with *human sustainability*, which requires maintaining human quality of life, rather than *green sustainability,* which may be defined as keeping the quality of the biosphere constant.[6] This objective can be justified by appealing to the Maximin principle, see Roemer (1998, 2007). It can be argued, and this is Rawls's position when justifying the (contemporaneous) "difference principle," that it is the quality of life of each person that should enter the maximin calculus, rather than subjective utility, which generally includes the satisfaction that the individual derives from the welfare of other people, such as her children.

Maximizing the QuoL of the worst-off generation will often require the maximization of the QuoL of the first generation subject to maintaining that QuoL for all future generations, so that there is no *QuoL growth* after the first generation.[7] Formally, the optimization program is of the following type.

Maximin Program: max Λ subject to $(c_t)^{\alpha_c}(x_t^l)^{\alpha_l}(S_t^n)^{\alpha_n}(\hat{S}^m - S_t^m)^{\alpha_m} \geq \Lambda, \quad t \geq 1,$

and subject to the feasibility conditions given by specific production relations, laws of motion of the stocks and resource constraints, and with the initial conditions given by the relevant stock values in the base year (2000).

At a solution of the Maximin Program, the path of the QuoL will typically be stationary, and it can be (at least asymptotically) supported by stationary paths in all the arguments of the QuoL function.

The Maximin Program models sustainability in our sense. Alternatively, the planner may seek a positive rate of growth in the QuoL of future generations at the cost of reducing the QuoL of Generation 1. It is, however, not obvious how to justify sacrifices of the worst-off present generation for the sake of improving the already higher quality of life of future ones.[8]

One might argue that parents want their children to have a higher quality of life than they do. Thus, growth of the quality of life might be supported by all parents over the maximin solution. An alternative justification for altruism towards future generations would appeal to *growth as a public*

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[6] See Eric Neumayer (1999) and the articles collected in Geir Asheim (2007) for the analysis of the various notions of sustainability.

[7] But not always: see Silvestre (2002).

[8] Recall that we assume away intragenerational inequality, thereby depriving economic growth of a role in alleviating contemporaneous poverty. This important topic has high priority in our research agenda.

good: we may feel justifiably proud of mankind's recent gains in, say, extraterrestrial travel, or average life expectancy, and wish them to continue into the far future even at a personal cost.[9]

Indeed, there is an asymmetry in the way we feel about contemporaneous vs. temporally disjoint inequality: a person in a poor country may not wish to sacrifice her quality of life for the sake of improving that of a person in a *richer* country, while at the same time be willing to make some sacrifices for the welfare of unrelated, yet-to-be born individuals who will as a consequence be richer than she.

Assume that society wants to achieve a sustained rate ρ of growth in the future QuoL: instead of maximizing the QuoL of the worst-off generation, it aims at the maximization of the QuoL of the first generation, subject to the condition that the QuoL subsequently grows at a given rate ρ per generation. The optimization program then becomes:

<u>Sustainable Growth Optimization Program</u>

max Λ subject to: $(c_t)^{\alpha_c}(x_t^l)^{\alpha_l}(S_t^n)^{\alpha_n}(\hat{S}^m - S_t^m)^{\alpha_m} \geq (1+\rho)^{t-1}\Lambda,\ t \geq 1,$

for $\rho \geq 0$ given, and subject again to the feasibility and initial conditions.

Note that the Maximin Program can be written in this form by letting $\rho = 0$.

At a solution to this program, the QuoL grows at a constant rate, but it is impossible to have steady positive growth of all variables because of the finite capacity $\hat{S}^m$ of the biosphere.

2.3. Economic constraints

Feasible paths are characterized by *economic constraints* and by *environmental stock-flow relations*. We adopt the following economic constraints. Recall that t = 1, 2, …. is measured in generations (25 years).

$$f(x_t^c, S_t^k, S_t^n, e_t, S_t^m) \equiv k_1 (x_t^c)^{\theta_c}(S_t^k)^{\theta_k}(S_t^n)^{\theta_n}(e_t)^{\theta_e}(S_t^m)^{\theta_m} \geq c_t + i_t,\ t \geq 1,$$

with $\theta_c > 0$, $\theta_k > 0$, $\theta_n > 0$, $\theta_c + \theta_k + \theta_n = 1$, $\theta_e > 0$, $\theta_m < 0$,

(Aggregate production function f)

$(1-\delta^k)S_{t-1}^k + k_2 i_t \geq S_t^k$, $t \geq 1$, (Law of motion of physical capital)

$(1-\delta^n)S_{t-1}^n + k_3 x_t^n \geq S_t^n$, $t \geq 1$, (Law of motion of the stock of knowledge)

$x_t^e + x_t^c + x_t^n + x_t^l \equiv x_t$, $t \geq 1$, (Allocation of efficient units of labor)

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[9] See Silvestre (2007).

$k_4 x_{t-1}^e \geq x_t$, $t \geq 1$, (Education production function)

with initial conditions (x_0^e, S_0^k, S_0^n), where c_t , x_t^l , S_t^n and S_t^m have been defined in Section 2.1 above, and where:

x_t^c = average annual efficiency units of labor per capita devoted to the production of output by Generation *t*;

e_t = average annual emissions of CO_2 in GtC by Generation *t*.

S_t^k = capital stock per capita available to Generation *t*;

i_t = average annual investment per capita by Generation *t*;

x_t^n = average annual efficiency units of labor per capita devoted to the production of knowledge by Generation *t*,

x_t^e = average annual efficiency units of labor per capita devoted to education by Generation *t*;

x_t = average annual efficiency units of time (labor and leisure) per capita available to Generation *t*.

We call emissions e_t and concentrations S_t^m *environmental variables*, whereas the remaining variables are called *economic*.

The following remarks compare our technology to some of those postulated in the growth literature.

Remark 1. The labor input in production, x_t^c, is measured in efficiency units of labor, which may be viewed as the number of labor-time units ("hours") multiplied by the amount of human capital embodied in one labor-time unit (as is customary since Hirofumi Uzawa, 1965 and Robert Lucas, 1988). Hence, because we assume that $\theta_c + \theta_k + \theta_n = 1$, our production function displays decreasing returns to "capital" when construed to consist of physical and human capital. But returns would be constant if we broadened the notion of "capital" to include also the stock of knowledge.

Remark 2. We assume that the production of new knowledge requires only efficiency labor (dedicated to R&D, or to "learning by not doing"), but that knowledge depreciates at a positive rate. These assumptions are in line with a large segment of the growth literature.

Remark 3. Our education production function, $x_t = k_4 x_{t-1}^e$, states that the education of a young generation requires only efficiency labor of the previous generation. If we normalize to unity the total labor-leisure time available to Generation *t*, then x_t can be interpreted as the amount of human capital per time unit in Generation *t*. Because our model is generational (*t* is a generation), instead of being an infinitely lived consumer (for whom *t* is just a moment in her life), our education production function cannot be interpreted in exactly the same manner as in many existing models of investment in human capital, which, in addition, are often cast in continuous time. More specifically, our formulation displays the following features.

(a) As in Uzawa (1965) and Lucas (1988), we do not include physical capital as an input in the production of education. This contrasts with Sergio Rebelo (1991) and Robert Barro and Xavier Sala-i-Martin (1999, p. 179). In the notation and wording of Barro and Sala-i-Martin, their "human capital production function" is

$$\dot{H} = B[(1-v)K]^{\eta}[(1-u)H]^{1-\eta} - \delta H , \qquad (1)$$

where H is the amount of human capital, (1 - v) K is the amount of physical capital used in education, (1 - u) is the fraction of human capital used in education, and B, η, and δ are parameters, the last one being the human-capital depreciation factor.

(b) We interpret the labor input in the production of education as that of teachers, rather than students. This departs from the interpretations by Lucas (1988) and Rebelo (1991), but it agrees with the comments in Uzawa (1965) and Barro and Sala-i-Martin (1999), e. g., the latter write (p. 179) "…a key aspect of education [is that] it relies heavily on educated people as an input."

(c) We see the education of a generation as a social investment, in line with Lucas's (1988, p. 19) dictum "…a general fact that I will emphasize again and again: that human capital accumulation is a *social* activity, involving *groups* of people, in a way that has no counterpart in the accumulation of physical capital." Also, we adopt a broad view of educational achievement, which in particular bestows the ability to adapt to new technologies, as emphasized by Claudia Goldin and Lawrence Katz (2008).

(d) Our education production function can be viewed as a generational version of (1) for the parameter values η = 0 and δ = 1 (since, in our model, all adults die at the end of each date), obtaining:

$$H_t - H_{t-1} = B[(1-u)H_{t-1}] - H_{t-1}, \text{ i. e., } H_t = B[(1-u)H_{t-1}] \qquad ,$$

which is precisely our education production function under the notational correspondence $H_t \leftrightarrow x_t$, $(1-u) \leftrightarrow \dfrac{x_{t-1}^e}{x_{t-1}}$ and $B \leftrightarrow k_4$.

2.4. Environmental stocks and flows

Anthropogenic greenhouse gas (GHG) emissions have caused atmospheric concentrations with no precedents in the last half a million years (see, e. g., Pierre Friedlingstein and Susan Salomon, 2005). The unparalleled behavior of GHG concentrations has motivated a growing literature that tries to predict the relationship among the paths of emissions, concentrations and global temperature changes.

Following a large segment of literature, we focus on CO_2 emissions and concentrations.[10] Recent climate research has revised upwards the persistence of the effects of GHG emissions. Haaron Kheshgi, Steven Smith and James Edmonds (2005, p. 213) emphasize that emitted CO_2 "is not destroyed in the atmosphere, but redistributed amongst the reservoirs that actively exchange carbon: plants and soils, oceans and the atmosphere." They argue that "for CO_2 to approach a constant concentration over finite time, CO_2 emissions must peak and then gradually approach zero over 1,000+ years, regardless of the concentration level." Alvaro Montenegro *et al*. (2007, p.1) argue that "higher levels of atmospheric CO_2 remain in the atmosphere than predicted by previous experiments, and the average perturbation lifetime of emissions is much longer than the 300-400 years proposed by other studies." Based on new evidence on the behavior of ocean temperatures after increases in emissions, H. Damon Matthews and Ken Caldeira (2008) show that temperatures will be rising long after the CO_2 concentration in the atmosphere has been stabilized and that in order "to achieve atmospheric carbon dioxide levels that lead to climate stabilization, the net addition of CO_2 to the atmosphere from human activities must be decreased to nearly zero." Similar conclusions are reached by Friedlingstein and Solomon (2005).

Most of the more recent and detailed physical models have no steady states, in the strict sense, with positive emissions. But if emissions are steady at low enough levels, then the stock of GHG eventually grows very slowly, experiencing minor increases in a scale of thousands of years. The effects of climate change on human welfare can then be substantially attenuated via mitigation (e. g., the construction of levees) and adaptation (e. g., moving North). The stocks of GHG are then said to be "stabilized" even though, strictly speaking, they are not constant in the very long run. Here we assume a constant "long term" value of the stock of GHG, where "constant" is a simplification of "stabilized," and where the "long term" scale refers to a few hundreds, but not thousands, of years.

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[10] The long-term effects of non-CO_2 GHG emissions have been addressed by Marcus Sarofim *et al*. (2005).

2.5. Our postulated path of GHG emissions

Because of the complexity of the climate models proposed and the lack of a canonical physical model of the current state of climatology, we do not attempt to specify the set of feasible flow-stock sequences $((e_t, S_t^m))_{t=1}^{\infty}$ and, accordingly, we do not try to compute optimal paths for emissions and the environmental stock. Instead, we adopt a simple path inspired by Meehl *et al*. (2007, Section 10.4), in particular by emission paths that lead to relatively low levels of stabilized concentrations of CO_2 under the assumption of coupling between climate change and the carbon cycle.[11] We choose the target stabilization level of 450 ppm (Meehl *et al*., 2007, Section 10.4, Figure 10.21(a)) and, conservatively, the path of coupled emissions for the Hadley model, as in C. D. Jones *et al*. (2006) (Figure 10.21.(c)).

These paths involve increasing emissions in the near future, and drastically reduced emissions in the more distant future. We adopt this general pattern, but we simplify the path by postulating only three levels of emissions and stock, which average over each generation the abovementioned lifetime paths for emissions, while taking as stock values those dated at the end of the life of the generation. Hence, the Meehl *et al*. (2007) analysis justifies the feasibility of our paths given the initial values $(\bar{e}_{2000}, \bar{S}^m_{2000}) = (6.58, 177.1)$ at year 2,000. The postulated (emission, concentration) pairs are:

$$(e_1, S_1^m) = (6.97, 303) \text{ for Generation 1,}$$

$$(e_2, S_2^m) = (4.43, 354) \text{ for Generation 2,}$$

and $$(e_t, S_t^m) = (e^*, S^{m*}) = (0.4, 363) \text{ for Generation } t,\ t \geq 3,$$

see Figure 2 in Section 6.1 below for a graphical representation.

Our choices for (e_1, S_1^m), (e_2, S_2^m) and (e^*, S^{m*}) imply that, in 2075, the concentration of CO_2 in the atmosphere is of 450 ppm (this corresponds to our value of $S^m* = 363$ GtC in the atmospheric stock of CO_2 beyond the preindustrial stock, see footnote 5 above). The algorithm described in Section 3.3 below motivates our choice of a two-generation interval to reach the target stabilization level.

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[11] The growth of the atmospheric CO_2 induces a climate change that affects the carbon cycle. In their words (p. 789) "There is an unanimous agreement among the models that future climate change will reduce the efficiency of the land and ocean carbon cycle to absorb anthropogenic CO_2, essentially owing to a reduction in the land carbon uptake."

The path that we choose entails, relative to the 1990 world emission levels, a 24% reduction in 2025 and a 93% reduction in 2050. For comparison, the American Clean Energy and Security Act HR 2454 of 2009 (US House of Representatives, Waxman-Markey Bill) aims at a 1% reduction in 2020 and an 80% reduction in 2050.[12] And the United Nations Human Development Report 2007/2008 (Overview, p. 29) recommends, for developed countries, between a 20% and a 30% for 2020, and an 80% for 2050.

2.6. Calibrated values

Appendix 3 below details our calibration procedures, which yield the following values.

Parameter	Value
α_c	0.32
α_l	0.65
α_n	0.02
α_m	0.01
k_1	16.408
k_2	13.118
k_3	649.34
k_4	35.45
θ_c	0.67
θ_k	0.28
θ_n	0.06
θ_m	-0.0075
θ_e	0.091
δ^k	0.787
δ^n	0.787
$\hat{S}^m$	781.55

Table 1. Calibrated parameter values

[12] The act targets take 2005 as the reference year. From the CAIT, World Resources Institute, we take US emissions in 2005 to be 20% higher than in 1990.

Stocks	Value	Units
$\bar{S}^k_{2000}$	73.65	thousands of 2000 dollars per capita.
$\bar{S}^n_{2000}$	15.64	thousands of 2000 dollars per capita.
$\bar{S}^m_{2000}$	177.1	GtC above pre-industrial level.
$\bar{x}_{2000}$	1.396	1950-efficiency units per capita.

Flows	Value	Units
$\bar{x}^e_{2000}$	0.0465	1950-efficiency units per capita.
$\bar{c}_{2000}$	23.88	thousands of 2000 dollars per capita.
$\bar{i}_{2000}$	7.59	thousands of 2000 dollars per capita.
$\bar{e}_{2000}$	6.58	GtC.
$\bar{y}_{2000}$	34.78	thousands of 2000 dollars per capita.

Table 2. Initial values in the benchmark year (2000)

3. Computational strategy and algorithm

Our computational strategy is based on the Ray Optimization Theorem below, in the spirit of turnpike theory: see our companion paper Llavador *et al*. (2009) for a turnpike theorem in a simpler model. Consider a pair (e^*, S^{m*}) such that the constant sequence $((e^*, S^{m*}))_{t=1}^{\infty}$ is an environmentally feasible flow-stock path, and the following optimization program.

Program $E[\rho, e^*, S^{m*}]$. Given (ρ, e^*, S^{m*}), Max Λ_1 subject to

$$\begin{aligned}
& c_t^{\alpha_c}(x_t^l)^{\alpha_l}(S_t^n)^{\alpha_n}(\hat{S}^m - S^{m*})^{\alpha_m} \geq \Lambda_1(1+\rho)^{t-1}, t \geq 1, \\
& k_1(x_t^c)^{\theta_c}(S_t^k)^{\theta_k}(S_t^n)^{\theta_n}(e^*)^{\theta_e}(S^{m*})^{\theta_m} \geq c_t + i_t, \quad t \geq 1, \\
& (1-\delta^k)S_{t-1}^k + k_2 i_t \geq S_t^k, \quad t \geq 1, \\
& (1-\delta^n)S_{t-1}^n + k_3 x_t^n \geq S_t^n, \quad t \geq 1, \\
& x_t^e + x_t^n + x_t^l + x_t^c \equiv x_t, \quad t \geq 1, \\
& k_4 x_{t-1}^e \geq x_t, \quad t \geq 1,
\end{aligned}$$

with initial conditions (x_0^e, S_0^k, S_0^n).

Recall that ρ is the rate of growth of the QuoL per generation. It will be convenient to denote by g the rate of growth of the economic variables, again per generation.

Theorem 1: Ray Optimization Theorem . *Assume constant returns to scale in production in the sense that* $\theta_c + \theta_k + \theta_n = 1$. *Given* $(g, e^*, S^{m*}) \in [0, k_4 - 1] \times \Re_{++} \times (0, \hat{S}^m)$, *there is a ray* $\Gamma(g, e^*, S^{m*}) \equiv \{(x^e, S^k, S^n) \in \Re^3_+ : (S^k, S^n) = x^e(q^k(g, e^*, S^{m*}), q^n(g))\}$, *such that if* $(x_0^e, S_0^k, S_0^n) \in \Gamma(g, e^*, S^{m*})$, $(x_0^e, S_0^k, S_0^n) \neq 0$, *then the solution path to Program* $E[\rho, e^*, S^{m*}]$ *satisfies:*

(i) $(x_t^e, S_t^k, S_t^n) = (1+g)^t (x_0^e, S_0^k, S_0^n)$, $t \geq 1$, *and hence* $(x_t^e, S_t^k, S_t^n) \in \Gamma(g, e^*, S^{m*})$, $t \geq 0$;

$$c_1 = p^c(g) q^k(g, e^*, S^{m*}) x_0^e,$$
$$i_1 = p^i(g) q^k(g, e^*, S^{m*}) x_0^e;$$

(ii) $x_1^l = \nu^l(g) q^n(g) x_0^e$,

$$x_1^n = \nu^n(g) q^n(g) x_0^e,$$
$$x_1^c = \nu^c(g) q^n(g) x_0^e;$$

(iii) $(c_t, i_t, x_t^l, x_t^n, x_t^c) = (1+g)^{t-1} (c_1, i_1, x_1^l, x_1^n, x_1^c)$, $t \geq 1$.

The quality of life grows at rate ρ, *were* $1 + \rho = (1+g)^{1-\alpha_m}$, *and all other variables grow at rate g, except for emissions and concentrations, which remain constant at* (e^*, S^{m*}).

Proof. Appendix 1, where the various proportionality factors (q, p, ν) are computed in terms of the parameters of the model. Table 3 illustrates the theorem.

In particular, it is important to observe that, for $g = \rho = 0$, whenever the initial endowments (x_0^e, S_0^k, S_0^n) lie in $\Gamma(0, e^*, S^{m*})$, the solution to Program $E[0, e^*, S^{m*}]$ is stationary over time.

We conjecture that a turnpike theorem, analogous to the one in Llavador *et al*. (2009), is true for Program $E[\rho, e^*, S^{m*}]$ for any g, and so, if we begin with an endowment vector off the ray $\Gamma(g, e^*, S^{m*})$, then the optimal solution will converge to the ray $\Gamma(g, e^*, S^{m*})$. Hence, in the long run, the solution will be almost a steady-state path. Motivated by this conjecture, we now construct feasible paths which begin at the actual year-2000 endowment values $(\bar{x}_{2000}^e, \bar{S}_{2000}^k, \bar{S}_{2000}^n)$ and reach the ray $\Gamma(g, e^*, S^{m*})$ in two generations, taking as given the values (e^*, S^{m*}), (e_1, S_1^m) and (e_2, S_2^m) reported in 2.5 above.[13]

[13] Inspired by IPCC AR4 (2007), we have computed paths in which carbon concentrations converge to the stabilized level in two generations. However, our optimization program could be run for slower convergence paths, with

	STOCKS				FLOWS				QuoL
Initial Cond.	x_0^e	$S_0^k = q^k x_0^e$	$S_0^n = q^n x_0^e$						
$t = 1$	$x_1^e = (1+g)x_0^e$	$S_1^k = q^k x_1^e = (1+g)S_0^k$	$S_1^n = q^n x_1^e = (1+g)S_0^n$	S^{m*}	e^*	$c_1 = p^c q^k x_0^e$	$i_1 = p^i q^k x_0^e$	$x_1^j = \nu^j q^k x_0^e$, $j = l,n,c$	Λ_1
$t = 2$	$x_2^e = (1+g)^2 x_0^e$	$S_2^k = q^k x_2^e = (1+g)^2 S_0^k$	$S_2^n = q^n x_2^e = (1+g)^2 S_0^n$	S^{m*}	e^*	$c_2 = p^c q^k x_1^e = (1+g)c_1$	$i_2 = p^i q^k x_1^e = (1+g)i_1$	$x_2^j = \nu^j q^k x_1^e = (1+g)x_1^j$, $j = l,n,c$	$(1+\rho)\Lambda_1$
t	$x_t^e = (1+g)^t x_0^e$	$S_t^k = q^k x_t^e = (1+g)^t S_0^k$	$S_t^n = q^n x_t^e = (1+g)^t S_0^n$	S^{m*}	e^*	$c_t = p^c q^k x_{t-1}^e = (1+g)^{t-1} c_1$	$i_t = p^i q^k x_{t-1}^e = (1+g)^{t-1} i_1$	$x_t^j = \nu^j q^k x_{t-1}^e = (1+g)^{t-1} x_1^j$, $j = l,n,c$	$(1+\rho)^{t-1}\Lambda_1$

Table 3. Stocks and flows in Theorem 1.

More precisely, for various rates of growth $\rho \geq 0$ of the QuoL (or associated rates of growth g of the variables), we construct feasible paths (Λ_1, Λ_2,…) such that the ratio $\frac{\Lambda_t}{\Lambda_{t-1}}$ of quality-of-life growth experienced by the later generations $t \geq 2$ is $1 + \rho$, and analyze the implications of these sustained growth factors for the QuoL Λ_1 of Generation 1. A reference level of QuoL is the one determined by the year-2000 values of the relevant variables, to be denoted Λ_0.

We proceed in two steps. First, we solve the optimization problem for (endogenous) initial conditions guaranteeing that the optimal solution is a steady state (i. e., all economic variables, not

convergence in three or more generations, at the cost of additional complexity in computation. We have also tried to maximize the QuoL of Generation 1 subject to its reaching the ray. Note that Generation 1's investment in knowledge (which affects the QuoL of Generation 1 both directly and indirectly through production) and Generation 1's investment in physical capital (which affects the QuoL of Generation 1 only indirectly through production) create intergenerational public goods. It turns out that, even for a zero-growth target, when Generation 1 maximizes its own QuoL subject to the stock proportionality dictated by the ray, it invests so heavily as to make the QuoL of the future generations higher that its own, a feature formally similar to the one discussed in Silvestre (2002). The resulting path yields therefore an unnecessarily low value. It is for this reason that we choose Generation 2 as the first one that has stocks on the ray.

including the environmental ones, grow at the same, predetermined rate.) Second, we go from the historical initial conditions to the steady state path in two generations, while keeping the rate of growth of the QuoL for all generations after the first one at the predetermined rate.

The QuoL of Generation t is given by $c_t^{\alpha_c}(x_t^l)^{\alpha_l}(S_t^n)^{\alpha_n}(\hat{S}^m - S_t^m)^{\alpha_m}$. If all variables (except biospheric quality) grow at a rate g, then the QuoL will grow at rate ρ where $1+\rho = (1+g)^{1-\alpha_m}$. A balanced growth solution relative to our choice requires three growth rates:

g for the variables $(S^n, x^n, x^e, x^c, x^l)$,

γ for the variables i, c and S^k,

ρ for the QuoL.

But ρ and γ are functions of g: so there is one independently chosen growth rate. For $\theta_c + \theta_k + \theta_n = 1$, we have that $g = \gamma$.

We apply the following two-step algorithm for the chosen $(\rho, e_1, S_1^m, e_2, S_2^m, e^*, S^{m*})$.

Step 1. For an arbitrary x_2^e, solve the following program.

Program $G[x_2^e]$. Given $(\rho, e_1, S_1^m, e_2, S_2^m, e^*, S^{m*})$ and x_2^e, Max Λ_1 subject to

$$c_1^{\alpha_c}(x_1^l)^{\alpha_l}(S_1^n)^{\alpha_n}(\hat{S}^m - S_1^m)^{\alpha_m} \geq \Lambda_1,$$

$$c_2^{\alpha_c}(x_2^l)^{\alpha_l}(S_2^n)^{\alpha_n}(\hat{S}^m - S_2^m)^{\alpha_m} \geq (1+\rho)\Lambda_1,$$

$$(x_2^e, S_2^k, S_2^n) \in \Gamma(g, e^*, S^{m*}),$$

$$k_1(x_1^c)^{\theta_c}(S_1^k)^{\theta_k}(S_1^n)^{\theta_n}(e_1)^{\theta_e}(S_1^m)^{\theta_m} \geq c_1 + i_1,$$
$$k_1(x_2^c)^{\theta_c}(S_2^k)^{\theta_k}(S_2^n)^{\theta_n}(e_2)^{\theta_e}(S_2^m)^{\theta_m} \geq c_2 + i_2,$$
$$(1-\delta^k)S_0^k + k_2 i_1 \geq S_1^k,$$
$$(1-\delta^k)S_1^k + k_2 i_2 \geq S_2^k,$$
$$(1-\delta^n)S_0^n + k_3 x_1^n \geq S_1^n,$$
$$(1-\delta^n)S_1^n + k_3 x_2^n \geq S_2^n,$$
$$k_4 x_0^e \geq x_1^e + x_1^n + x_1^l + x_1^c$$
$$k_4 x_1^e \geq x_2^e + x_2^n + x_2^l + x_2^c,$$

for the initial conditions $(x_0^e, S_0^k, S_0^n) = (\bar{x}_{2000}^e, \bar{S}_{2000}^k, \bar{S}_{2000}^n)$: here and in what follows, the year-2000 value for a variable is indicated by an overbar and a 2000 subscript. See Section 2.6 above for year-2000 numerical values.

Table 4 illustrates Step 1 in our computation procedure.

Step 2. Note that the QuoL of Generation 3, and of all subsequent generations, is determined by x_2^e. By trial and error, we locate the value of x_2^e with the property that, at the solution to Program $G[x_2^e]$, the QuoL of Generation 3 equals $(1 + \rho)^2\Lambda_1$. Note that then the QuoL of Generation *t*, $t \geq 4$, is $(1 + \rho)^{t-3}$ times the QuoL of Generation 3 (by Theorem 1), and that, by the second constraint of Program $G[x_2^e]$, the QuoL of Generation 2 is $(1 + \rho)\Lambda_1$. Hence, the QuoL of Generation *t* is $(1+\rho)^{t-1}\Lambda_1$, for all $t \geq 1$.

Appendix 2 writes the solution to Program $G[x_2^e]$ as a system of 14 equations in the 14 endogenous variables $(\Lambda_1, c_1, x_1^l, x_1^c, x_1^n, x_1^e, c_2, x_2^l, x_2^c, x_2^n, i_1, i_2, S_1^k, S_1^n)$, which is then reduced to a system of seven equations in seven unknowns. Then, using *Mathematica*, we compute the numerical solution paths to Program $G[x_2^e]$ for our calibrated parameter values, and adjust x_2^e so that the QuoL of Generation 3 equals $(1 + \rho)^2\Lambda_1$, implying, as noted above, that the QuoL of Generation *t* is $(1 + \rho)^{t-1}\Lambda_1$, for all $t \geq 1$. We perform this calculation for three sustained growth rates of the QuoL, namely $\hat{\rho} = 0.00$ (no growth), $\hat{\rho} = 0.01$ and $\hat{\rho} = 0.02$, where $\hat{\rho}$ is the rate of growth of the QuoL expressed in *per annum* terms, with corresponding rates of growth per generation (defined by $\rho = (1+\hat{\rho})^{25}$) equal to $\rho = 0.00$, $\rho = 0.28$ and $\rho = 0.64$, respectively.

	STOCKS				FLOWS				QuoL
Year 2000	$\bar{x}^e_{2000}$	$\bar{S}^k_{2000}$	$\bar{S}^n_{2000}$	$\bar{S}^m_{2000}$	$\bar{e}_{2000}$	$\bar{c}_{2000}$	$\bar{i}_{2000}$	$\bar{x}^j_{2000}$, $j = l,n,c$	Λ_0
t=1	x_1^e	S_1^k	S_1^n	S_1^m	e_1	c_1	i_1	x_1^j, $j = l,n,c$	Λ_1
t=2	x_2^e	S_2^k $= q^k x_2^e$	S_2^n $= q^n x_2^e$	S_2^m	e_2	c_2	i_2	x_2^j, $j = l,n,c$	$(1+\rho)\Lambda_1$
t=3	x_3^e $= (1+g)x_2^e$	S_3^k $= q^k x_3^e$ $= (1+g)S_2^k$	S_3^n $= q^n x_3^e$ $= (1+g)S_2^n$	S^{m*}	e^*	c_3 $= p^c q^k x_2^e$	i_3 $= p^i q^k x_2^e$	x_3^j $= \nu^j q^k x_2^e$, $j = l,n,c$	Λ_3
t=4	x_4^e $= (1+g)^2 x_2^e$	S_4^k $= q^k x_4^e$ $= (1+g)^2 S_2^k$	S_4^n $= q^n x_4^e$ $= (1+g)^2 S_2^n$	S^{m*}	e^*	c_4 $= p^c q^k x_3^e$ $= (1+g)c_3$	i_4 $= p^i q^k x_3^e$ $= (1+g)i_3$	x_4^j $= \nu^j q^k x_3^e$ $= (1+g)x_3^j$, $j = l,n,c$	$(1+\rho)\Lambda_3$
$t \geq 4$	x_t^e $= (1+g)^{t-2} x_2^e$	S_t^k $= q^k x_t^e$ $= (1+g)^{t-2} S_2^k$	S_t^n $= q^n x_t^e$ $= (1+g)^{t-2} S_2^n$	S^{m*}	e^*	c_t $= p^c q^k x_{t-1}^e$ $= (1+g)^{t-3} c_3$	i_t $= p^i q^k x_{t-1}^e$ $= (1+g)^{t-3} i_3$	x_t^j $= \nu^j q^k x_{t-1}^e$ $= (1+g)^{t-3} x_3^j$, $j = l,n,c$	$(1+\rho)^{t-3}\Lambda_3$

Table 4. Step 1 in our computation procedure, where $q^k = q^k(g, e^*, S^{m*}), q^n = q^n(g), v^j = v^j(g)(j = l,n,c), p^j = p^j(g)(j = c,i)$.

4. Results

Tables 5-7 describe the obtained paths. The first rows in the tables display the year-2000 values, repeated in each table to facilitate comparison. Some of the information in these tables is summarized in Tables 8-10 and depicted in Figure 1. Recall (see Section 2.5 above) that we postulate a rather conservative path of GHG emissions aimed at stabilizing GHG concentrations at a moderate value, i. e., $(e_1, S_1^m) = (6.97, 303)$, $(e_2, S_2^m) = (4.43, 354)$, $(e_t, S_t^m) = (e^*, S^{m*}) = (0.4, 363), t \geq 3$.

Gen	$\frac{\Lambda_t}{\Lambda_0}$	$\frac{\Lambda_t}{\Lambda_{t-1}}$	c_t	$\frac{c_t}{c_0}$	$\frac{c_t}{c_{t-1}}$	i_t	S_t^k	S_t^n
2000	1.	-	23.88	1.	-	7.59	73.65	15.64
1	1.4011	1.4011	49.562	2.0755	2.0755	17.79	249.04	49.00
2	1.4011	1.	46.185	1.9340	0.9319	9.37	176.02	54.70
3	1.4011	1.	37.074	1.5525	0.8027	10.56	176.02	54.70
4	1.4011	1	37.074	1.5525	1	10.56	175.73	54.70

Gen	x_t	x_t^e	x_t^c	x_t^n	x_t^l	x_t^e (%)	x_t^c (%)	x_t^n (%)	x_t^l (%)
2000	1.396	0.0465	0.3955	0.0233	0.9307	0.0333	0.2833	0.0167	0.6667
1	1.648	0.0465	0.4741	0.0703	1.0576	0.0282	0.2876	0.0427	0.6416
2	1.647	0.0523	0.4338	0.0682	1.0932	0.0317	0.2633	0.0414	0.6636
3	1.853	0.0523	0.5164	0.0663	1.2185	0.0282	0.2786	0.0358	0.6574
4	1.853	0.0523	0.5164	0.0663	1.2185	0.0282	0.2786	0.0358	0.6574

Table 5. $\hat{\rho} = 0.00$ (sustainable QuoL, no growth)

Gen	$\frac{\Lambda_t}{\Lambda_0}$	$\frac{\Lambda_t}{\Lambda_{t-1}}$	c_t	$\frac{c_t}{c_0}$	$\frac{c_t}{c_{t-1}}$	i_t	S_t^k	S_t^n
2000	1.	-	23.880	1.	-	7.59	73.65	15.64
1	1.3892	1.389	49.135	2.0576	2.0576	17.62	246.84	48.57
2	1.7816	1.282	58.930	2.4678	1.1994	13.42	228.67	70.32
3	2.2848	1.282	60.860	2.5486	1.0328	18.70	293.99	90.41
4	2.9301	1.282	78.246	3.2766	1.2857	24.04	377.97	116.23

Gen	x_t	x_t^e	x_t^c	x_t^n	x_t^l	x_t^e (%)	x_t^c (%)	x_t^n (%)	x_t^l (%)
2000	1.396	0.0465	0.3955	0.0233	0.9307	0.0333	0.2833	0.0167	0.6667
1	1.648	0.0602	0.4699	0.0697	1.0486	0.0365	0.2851	0.0423	0.6361
2	2.136	0.0870	0.5639	0.0924	1.3924	0.0407	0.2640	0.0432	0.6520
3	3.083	0.1118	0.8600	0.1162	1.9946	0.0363	0.2790	0.0377	0.6471
4	3.963	0.1437	1.1057	0.1493	2.5644	0.0363	0.2790	0.0377	0.6471

Table 6. $\hat{\rho} = 0.01$ (1% annual growth or 28% generational growth)

Gen	$\frac{\Lambda_t}{\Lambda_0}$	$\frac{\Lambda_t}{\Lambda_{t-1}}$	c_t	$\frac{c_t}{c_0}$	$\frac{c_t}{c_{t-1}}$	i_t	S_t^k	S_t^n
2000	1.	1.	23.880	1.		7.59	73.65	15.64
1	1.3742	1.3742	48.592	2.0348	2.0348	17.41	244.04	48.03
2	2.2545	1.6406	74.850	3.1344	1.5404	18.49	294.54	89.77
3	3.6987	1.6406	99.184	4.1534	1.3251	32.24	485.64	148.02
4	6.0681	1.6406	163.538	6.8683	1.6488	53.16	800.74	244.06

Gen	x_t	x_t^e	x_t^c	x_t^n	x_t^l	x_t^e (%)	x_t^c (%)	x_t^n (%)	x_t^l (%)
2000	1.396	0.0465	0.3955	0.0233	0.9307	0.0333	0.2833	0.0167	0.6667
1	1.648	0.0778	0.4647	0.0688	1.0372	0.0472	0.2819	0.0418	0.6292
2	2.757	0.1437	0.7258	0.1225	1.7646	0.0521	0.2633	0.0444	0.6401
3	5.093	0.2369	1.4166	0.1985	3.2412	0.0465	0.2781	0.0390	0.6364
4	8.398	0.3906	2.3357	0.3273	5.3441	0.0465	0.2781	0.0390	0.6364

Table 7. $\hat{\rho} = 0.02$ (2% annual growth or 64% generational growth)

Our computations yield the following results. They are expressed in levels and rates of growth of the QuoL, but they also hold (with some modifications for generations 1, 2 and 3) if we more narrowly focus on the levels and growth of, say, consumption: see the columns labeled $\frac{c_t}{c_0}$ and $\frac{c_t}{c_{t-1}}$ in the tables.

(1) ***Human quality of life can be sustained forever at a level* 40%** ***higher than the year-*** **2000** ***reference level.***

See the first column of Table 5. The QuoL of the first generation jumps to 40% above that of the year-2000 reference level, and stays there forever. This fact is illustrated by the two horizontal lines in Figure 1: the lower, dotted line, with ordinate equal to 1, corresponds to the year-2000 reference level, while the continuous horizontal line with circular dots gives the sustained level of QuoL for all generations $t \geq 1$.

(2) ***Moderate growth rates can be achieved at the cost of a small reduction in the QuoL of the first generation, which stays well above the year*** **2000** ***reference level.***

A tradeoff between the QuoL of the first generation and the subsequent growth rates must indeed be expected. But our analysis shows that its magnitude is quite small: Generation 1's sacrifice for the sake of a higher growth rate is tiny for reasonable growth rates.

Table 8 (obtained from Tables 6 and 7) displays the relevant magnitudes. As just noted, human QuoL can be sustained forever while the QuoL of the first generation is 1.401 times the year 2000 reference level. The second row of Table 8 shows that, in order to subsequently maintain a 1% growth rate per year (28% per generation), the QuoL of the first generation would instead be 1.389, about 0.86% lower than the no-growth value. In other words, a maintained growth rate of 28% per generation can be reached at the cost of a less than 1% reduction of the QuoL of the first generation relative to the sustainable (no growth) path.

Similarly, the third row of Table 8 shows that in order to subsequently maintain a 2% growth rate per year (64% per generation), the QuoL of the first generation would be about 1.93 % lower than the sustainable, no-growth value. In other words, a maintained growth rate of 64 % per generation can be reached at the cost of a less than 2% reduction of the QuoL of the first generation relative to the no growth path.

	$\frac{\tilde{\Lambda}_1(\hat{\rho})}{\Lambda_0}$	$\frac{\tilde{\Lambda}_1(0)-\tilde{\Lambda}_1(\hat{\rho})}{\tilde{\Lambda}_1(0)}$
$\hat{\rho} = 0.00$ (Sustainable, No growth)	1.401	0.000
$\hat{\rho} = 0.01$ $\rho = 0.28$	1.389	0.0086 = 0.86%
$\hat{\rho} = 0.02$ $\rho = 0.64$	1.374	0.0193 = 1.93%

Table 8. The QuoL of the first generation (first column) relative to the year-2000 reference level Λ_0, and the sacrifice of the first generation to sustain subsequent positive growth rates (second column). The tildes denote the solution for the corresponding variable as a function of $\hat{\rho}$.

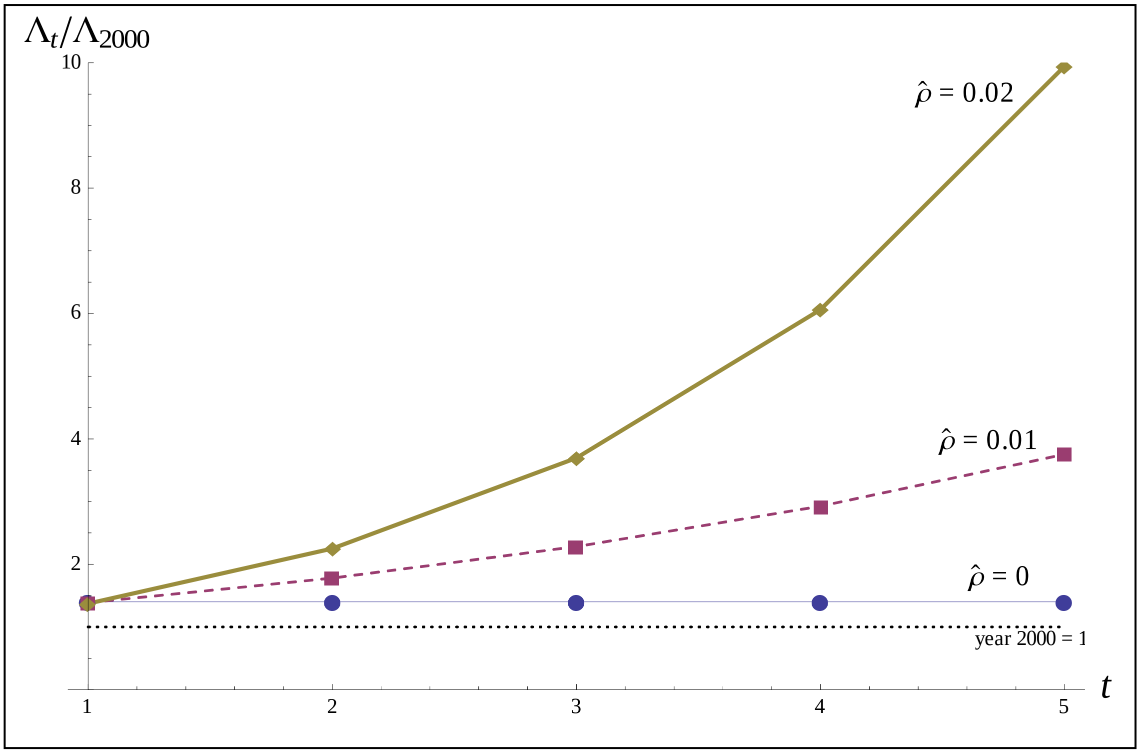

Figure 1.
Paths of quality of life of several generations for alternative rates of growth $\hat{\rho}$ (growth *per annum* in the Quality of Life).
All variables grow at a rate slightly higher than $\hat{\rho}$, with the exception of emissions and the stock of the biosphere, which follow the path described above, with constant low emissions and constant biosphere stock for $t \geq 3$.

Figure 1 shows the paths of the QuoL under the different growth targets. Note that they stay well above the year-2000 reference level. It is not possible at the scale of the graph to distinguish among the three values of the QuoL of the first generation (for annual growth rates of 0, 1% and 2%, respectively), all clustered close to the 1.3 value.

How are these QuoL paths achieved? Labor time is, in the reference year 2000, allocated to the various ends as follows (see, e. g., Table 5):

Fraction allocated to education:	0.0333
Fraction allocated to the creation of knowledge:	0.0167
Fraction allocated to investment in physical capital:	0.0683
Fraction allocated to the production of consumption:	0.2150
Fraction allocated to leisure:	0.6667
	1.0000

Table 9 indicates how these fractions should be modified in the proposed solutions. We observe the following features.

(3) ***The most important change required by the implementation of the desirable paths is the (more than) doubling the reference fraction of labor devoted to the creation of knowledge, whereas the fractions of labor allocated to consumption and leisure are similar to those of the reference year* 2000*.***

The largest change displayed in Table 9 occurs in the fraction allocated to knowledge, which must be about twice (2.14, 2.26 or 2.34) the year-2000 reference level. The fraction of labor time devoted to the production of the consumption good is slightly lower. We also observe a slight decrease in leisure time relative to the year 2000 reference values.

As might be expected, higher growth rates require higher fractions of labor dedicated to the various forms of investment (education, knowledge and physical capital), and lower fractions dedicated to consumption and leisure. But as just noted, the fractions of labor dedicated to consumption and leisure are not very sensitive to the growth rate, whereas it turns out that the fraction of labor devoted to education increases rapidly with the target growth rate. Table 10, obtained by dividing the values in rows 2 and 3 of Table 9 by those of the first row (and subtracting 1), illustrates. This yields the following result.

(4) ***Higher growth rates require substantial increases in the fraction of labor devoted to education (of the order of a* 30% *increase for each additional* 1% *of annual growth), together with moderate increases in the fractions of labor devoted to knowledge and the investment in physical capital (of the order of a* 5% *increase for each additional* 1% *of annual growth). Higher growth rates also require minor decreases in the amount of labor-time devoted to the production of consumption goods and to leisure (of the order of a* 2% *decrease for each additional* 1% *of annual growth).***

	Education	Knowledge	Investment in Phys. Capital	Consumption	Leisure
	$\dfrac{\tilde{x}_t^e(\hat{\rho})/\tilde{x}_t(\hat{\rho})}{\overline{x}_{2000}^e/\overline{x}_{2000}}$	$\dfrac{\tilde{x}_t^n(\hat{\rho})/\tilde{x}_t(\hat{\rho})}{\overline{x}_{2000}^n/\overline{x}_{2000}}$	$\dfrac{\frac{\tilde{i}_t(\hat{\rho})}{\tilde{c}_t(\hat{\rho})+\tilde{i}_t(\hat{\rho})}\frac{\tilde{x}_t^c(\hat{\rho})}{\tilde{x}_t(\hat{\rho})}}{\frac{\overline{i}_{2000}}{\overline{c}_{2000}+\overline{i}_{2000}}\frac{\overline{x}_{2000}^c}{\overline{x}_{2000}}}$	$\dfrac{\frac{\tilde{c}_t(\hat{\rho})}{\tilde{c}_t(\hat{\rho})+\tilde{i}_t(\hat{\rho})}\frac{\tilde{x}_t^c(\hat{\rho})}{\tilde{x}_t(\hat{\rho})}}{\frac{\overline{c}_{2000}}{\overline{c}_{2000}+\overline{i}_{2000}}\frac{\overline{x}_{2000}^c}{\overline{x}_{2000}}}$	$\dfrac{\tilde{x}_t^\ell(\hat{\rho})/\tilde{x}_t(\hat{\rho})}{\overline{x}_{2000}^\ell/\overline{x}_{2000}}$
$\hat{\rho}=0$ (No growth)	0.85	2.14	0.89	0.99	0.99
$\hat{\rho}=0.01$ $\rho=0.282$	1.09	2.26	0.94	0.97	0.97
$\hat{\rho}=0.02$ $\rho=0.64$	1.40	2.34	0.98	0.96	0.95

Table 9. Comparison between steady state and year-2000 values of the allocation of labor for the various growth rates. Again, the tildes denote the solution for the corresponding variable as a function of $\hat{\rho}$.

	Education	Knowledge	Investment in Phys. Capital	Consumption	Leisure
	$\dfrac{\tilde{x}_t^e(\hat{\rho})/\tilde{x}_t(\hat{\rho})}{\tilde{x}_t^e(0)/\tilde{x}_t(0)}-1$	$\dfrac{\tilde{x}_t^n(\hat{\rho})/\tilde{x}_t(\hat{\rho})}{\tilde{x}_t^n(0)/\tilde{x}_t(0)}-1$	$\dfrac{\frac{\tilde{i}_t(\hat{\rho})}{\tilde{c}_t(\hat{\rho})+\tilde{i}_t(\hat{\rho})}\frac{\tilde{x}_t^c(\hat{\rho})}{\tilde{x}_t(\hat{\rho})}}{\frac{\tilde{i}_t(0)}{\tilde{c}_t(0)+\tilde{i}_t(0)}\frac{\tilde{x}_t^c(0)}{\tilde{x}_t(0)}}-1$	$\dfrac{\frac{\tilde{c}_t(\hat{\rho})}{\tilde{c}_t(\hat{\rho})+\tilde{i}_t(\hat{\rho})}\frac{\tilde{x}_t^c(\hat{\rho})}{\tilde{x}_t(\hat{\rho})}}{\frac{\tilde{c}_t(0)}{\tilde{c}_t(0)+\tilde{i}_t(0)}\frac{\tilde{x}_t^c(0)}{\tilde{x}_t(0)}}-1$	$\dfrac{\tilde{x}_t^\ell(\hat{\rho})/\tilde{x}_t(\hat{\rho})}{\tilde{x}_t^\ell(0)/\tilde{x}_t(0)}-1$
$\hat{\rho}=0.01$ $\rho=0.28$	28 % above no growth	5 % above no growth	6 % above no growth	2% below no growth	2% below no growth
$\hat{\rho}=0.02$ $\rho=0.64$	65% above no growth	9 % above no growth	10 % above no growth	3% below no growth	3% below no growth

Table 10. The sensitivity of the fractions of labor resources devoted to each activity with respect to the target growth rate.

We have tested for the robustness of our results in several ways. In addition to our calibrated values, we also considered a lower $\hat{S}^m$ = 650 ppm and a higher $\hat{S}^m$ =750 ppm, as well as lower and higher values for $\frac{\alpha_m}{\alpha_c}$, θ_e, and θ_m, and we have obtained qualitatively similar results. Unsurprisingly, the sustainable level of human QuoL increases with the catastrophic level of carbon concentration in the atmosphere ($\hat{S}^m$) and with the elasticities of output to emissions (θ_e) and to carbon concentration (θ_m), and decreases with the relative weight of the environment in QuoL (α_m).Yet our qualitative conclusions continue to hold under these changes. Finally, we have also considered different values of parameters associated with the educational technology (k_4), and we have found that we can sustain forever levels of life-quality above the 2000 reference value, even for much lower values of k_4.[14]

5. Introducing uncertainty

A standard form of uncertainty in dynamic models concerns the date at which the human species will end. In this section, we refer to results from our companion paper (Llavador *et al*., 2009) that have a bearing on the analysis in this one.

We introduce the *discounted utilitarian* program associated with our problem. Denote by $\hat{P}[e*, S^{m*}]$ the set of feasible paths according to the constraints of Program $E[\rho, e^*, S^{m*}]$ of Section 3 above, for some fixed endowment vector (x_0^e, S_0^k, S_0^n). (This set is independent of the value of ρ.) The associated discounted-utilitarian program, with a discount factor of φ, is:

Program $DU[\varphi, e^*, S^{m*}]$:

$$\begin{aligned} &\max \sum_{t=1}^{\infty} \varphi^{t-1} \Lambda_t(\pi) \\ &\text{s.t. } \pi \in \hat{P}[e^*, S^{m*}] \end{aligned},$$

where $\Lambda_t(\pi)$ is the QuoL at date *t* along the path π. We have:

Corollary to Theorem 1. *Program* $DU[\varphi, e^*, S^{m*}]$ *diverges if* $\varphi k_4^{1-\alpha_m} > 1$.

Proof. By Theorem 1, for any $g < k_4 - 1$ there is a ray $\Gamma(g, e^*, S^{m*})$ such that, from any initial endowment vector on this ray, the balanced growth path where the economic variables grow at rate *g* is feasible. For any $g < k_4 - 1$, we can construct a path which, in a finite number of dates,

[14] For example, even for an unrealistically low value of k_4 equal to 30, human quality of life can be sustained forever at a level 18% higher than the year 2000 reference level.

moves from the given endowment vector (x_0^e, S_0^k, S_0^n) to some point on this ray. We then complete the path by appending the balanced growth path just referred to. Again by Theorem 1, the QuoL grows by a factor of $1+\rho$ at each date, after the initial section of the path, where $1+\rho=(1+g)^{1-\alpha_m}$. But g may be chosen so that $1+g$ is arbitrarily close to k_4. Hence, the terms of the discounted-utilitarian objective will grow by a factor arbitrarily close to $\varphi k_4^{1-\alpha_m}$: in particular, g can be chosen so that this factor is greater than one, by the premise, which proves the corollary. ■

It is notable that the 'power' of the technology, in the sense of whether or not Program $DU[\varphi, e^*, S^{m*}]$ diverges, depends only on the technological parameter k_4, associated with the educational technology, not on any parameters associated with the other two production functions. In a simpler model than the one here, studied in Llavador *et al.* (2009), we attempt to explain in an intuitive way why this is the case, and we shall not repeat that argument here. The fact depends upon the constant-returns technology, that labor is the single input in the production of skilled labor, and upon the constant-returns quality-of-life function. In particular, the last fact requires that leisure be measured in quality units, an assumption we strongly defend. As long as the assumption that the educational technology uses only educated labor as an input is approximately true, we believe this result is robust. We are reminded of Goldin and Katz (2008), who argue that the power of the American growth performance in the twentieth century was fundamentally due to universal education.

We suppose (following the Stern Review) that there is an exogenous probability *p* that mankind becomes extinct at any generation, and that there is an (independent) draw from this random variable at the end of each generation. To model the intergenerational welfare objective, we suppose that there is an Ethical Observer (EO) whose preferences satisfy the expected utility hypothesis. An outcome (or *prize*) is the event that mankind lasts exactly *T* generations, with a vector of qualities of life $(\Lambda_1, ..., \Lambda_T)$. The EO's von Neumann-Morgenstern utility at a given outcome is denoted $W^T(\Lambda_1, ..., \Lambda_T)$. This function, together with the $p(1-p)^{t-1}$ exogenous probability of extinction at (the end of) date t, define the expected utility of the EO when she chooses an infinite path $(\Lambda_1, \Lambda_2, ...)$ as $\sum_{t=1}^{\infty} p(1-p)^{t-1} W^t(\Lambda_1, ..., \Lambda_t)$.

A purely Rawlsian EO would only be concerned with the QuoL of the worst-off person who ever lived, and hence her vNM utility at the outcome $(\Lambda_1,...,\Lambda_T)$ would be $\min\{\Lambda_1,\Lambda_2,...,\Lambda_T\}$. More generally, we consider the family of vNM utility functions $[1+(T-1)\beta]\min\{\Lambda_1,\Lambda_2,...,\Lambda_T\}$ parameterized by $\beta \in [0,1]$, and we call an EO with such a vNM function an *extended Rawlsian EO*. Note that, for $\beta > 0$, such an EO takes into account both the QuoL of the worst-off generation and the future time span *T* of the human species.[15]

We prove in Llavador *et al*. (2009) that, for a simpler economy with education and physical capital as only intergenerational links, *if the discounted-utilitarian program with discount factor* $\varphi = 1-p$ *, diverges, then the solution to the optimization program of the Extended Rawlsian EO under uncertainty is exactly the solution to Program SUS.* In particular, Λ_t is constant with respect to *t*. The EO can then ignore the uncertainty!

We conjecture that an analogous result holds for the economy with the constraints of Program $E[\rho,e^*,S^{m^*}]$ of Section 3 above. If we take $1-p=0.975$ *per generation of* 25 *years*, as does the Stern Review, then, with the value of k_4 we have estimated, the discounted utilitarian program will diverge as long as $(1-p)k_4^{1-\alpha_m} > 1$, according to the Corollary proved above. But this inequality is surely true with our calibration of the parameters. Therefore we conjecture that the solution to the program of the Extended Rawlsian EO is just the solution to *SUS*, for the discount factor $\varphi = 1-p = 0.975$. The very rough intuition is that the possibilities for growth inherent in a large value of k_4 more than counteract the discount on the resources allocated to future generations that the EO might contemplate placing, due the possibility that they may not exist, if $(1-p)k_4 > 1$.

However, it must be remarked that a more important kind of uncertainty to introduce would be our uncertainty with regard to the physics of global warming. This is a more difficult undertaking.

6. Relation to the literature

6.1. Nordhaus's optimization

Nordhaus (2008a,b) proposes particular paths for CO_2 emissions, CO_2 concentrations and consumption per capita based on an optimization program with objective function

[15] We are indebted to Klaus Nehring for suggesting that we extend the pure Rawlsian EO to the "β = 1" case.

$$\sum_{t=1}^{T} L_t \frac{1}{1-\eta}(c_t)^{1-\eta}\frac{1}{(1+\delta)^t} \quad , \qquad (2)$$

where L_t is the number of people in generation t.[16] He calls the δ and η of (2) "central" and "unobserved normative parameters," reflecting "the relative importance of the different generations." (Nordhaus 2008a, p. 33, 60). Note that the maximin objective function of our Section 2.2 above could be viewed as a limit case of (2) for $L_t = 1$, $\delta = 0$ and $\eta \rightarrow \infty$. Nordhaus (2008a) chooses $\eta = 2$ and $\delta = (0.015)^{10}$, corresponding to a per year rate of $\hat{\delta} = 0.015$.[17] Appendix 4 below comments on Nordhaus's (2008a) objective function and on his calibration of its parameters.

The paths for emissions and concentrations proposed as optimal by Nordhaus differ markedly from the ones that we postulate: figures 2(a) (emissions) and 2(b) (concentrations) illustrate. Recall that we take as given a conservative path that drops emissions to very low levels by 2050 and stabilizes atmospheric CO_2 concentration at about 450 ppm by 2050. In striking contrast, Nordhaus (2008a, b) proposes as "optimal" a path where emissions and concentrations keep increasing past the end of the 21st century. Nordhaus (2008a, b) proposed values for 2100 are about 11 GtC in emissions, with concentrations at 586.4 ppm at 2100 and at a peak of about 680 ppm in 2180.

In light of the recent climate science research, we view Nordhaus's (2008a, b) "optimal" emission and concentration figures as excessively high, likely to bring about irreversible changes in temperature and unavoidable negative impacts in the welfare of future generations.

A striking feature of Nordhaus (2008a) is that the path for per capita consumption (his only variable in the individual utility function) is virtually identical (at least for the 21st century) in the "optimal" and in the "baseline" (*laissez faire*) paths, see his Figure 5.9. Yet he claims (p. 82) that the value of the objective function at the "optimal" solution is 3.37 trillions of 2005 US$ higher than at the baseline solution. We conjecture that this puzzle may be partially explained by population growth, which increases the value of the objective function for a given level of consumption per capita, together with minute differences in consumption per capita. Because of the little difference between the optimal and nonoptimal paths of consumption per capita, we conjecture that his rate of growth in consumption per capita is basically driven by his postulated exogenous growth in total factor productivity.

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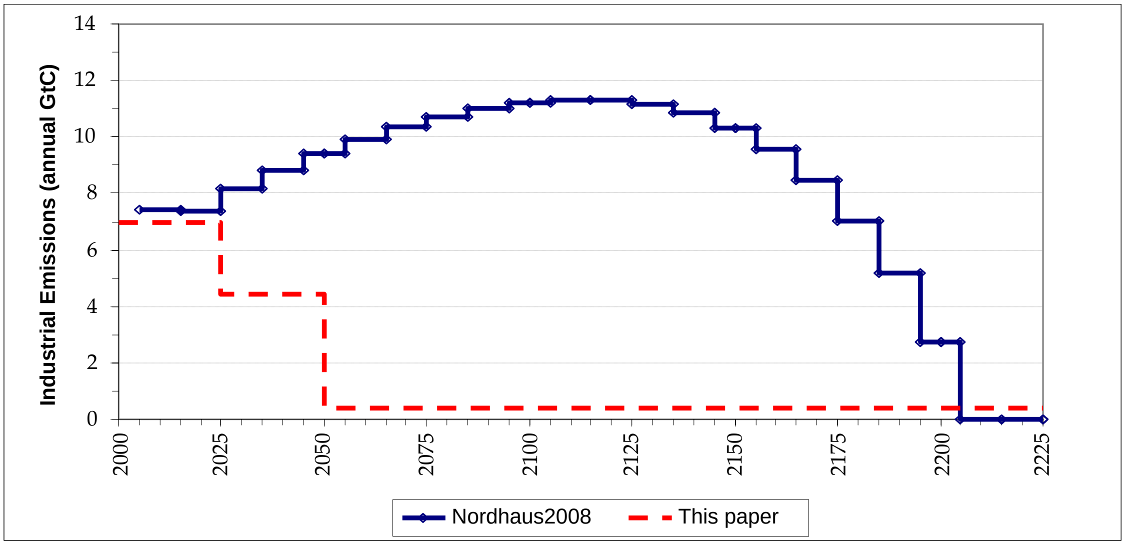
[16] The objective function is given in Nordhaus (2008a, p. 205), with each period $t = 1, 2, ...$ understood as a decade (instead of our 25-year generations). His notation is different. The optimization is numerically solved by the General Algebraic Modeling System (GAMS) program, see Nordhaus (2008b).

[17] The latter is half the value adopted in Nordhaus and Boyer (2000), see Nordhaus (2008a, p. 50).

2(a) CO_2 Emissions



2(b) CO_2 Concentrations

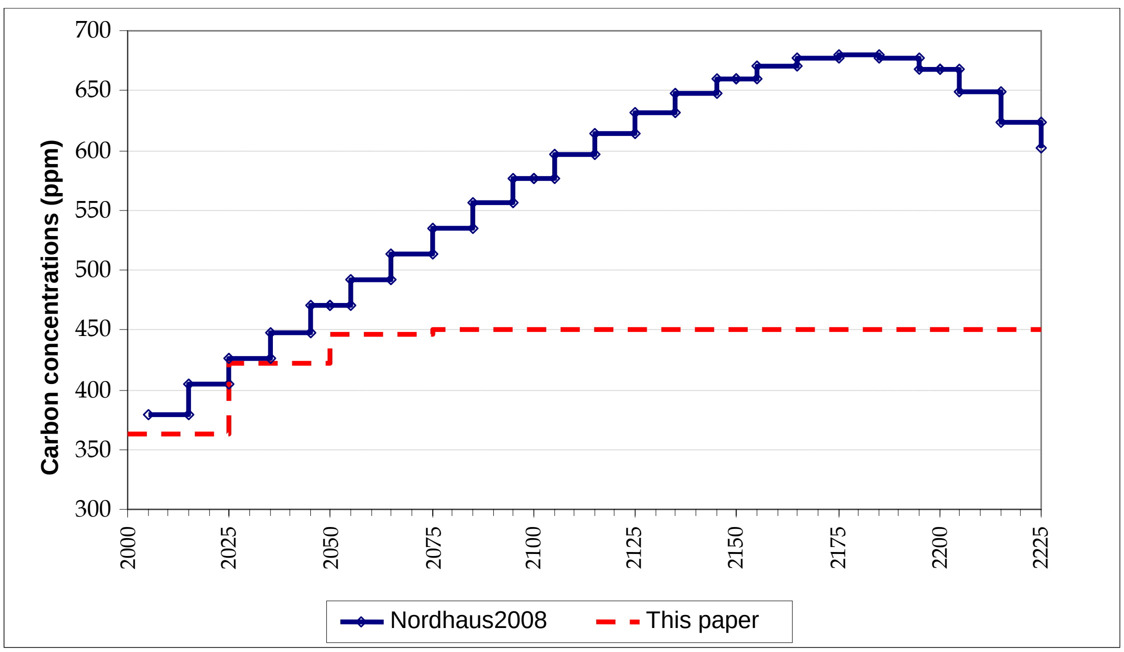


Figure 2.
Comparison of paths for the environmental variables proposed by Nordhaus (2008a,b) with the ones postulated in the present paper.

The paths for Nordhaus "Optimal" are computed by running the program GAMS with data provided in Nordhaus (2008b). The curve labeled "Optimal" of Figure 5-6 in Nordhaus (2008a) displays emissions only for the period 2005-2105, where they coincide with those of Figure 2(a) here (except that there the emissions are per decade, and here per year). Similarly, the curve labeled "Optimal" of Figure 5-7 in Nordhaus (2008a) displays concentrations only for the period 2005-2205, where they coincide with those of Figure 2(b) here.

6.2. Cost-Benefit analysis: The Stern Review

Cost-Benefit analysis underpins the recommendations of the Stern Review, in turn based on the Third Assessment Report of the United Nation's Intergovernmental Panel on Climate Change (TAR IPCC, 2001) and on Christopher Hope (2006). The Stern Review does not attempt to solve an optimization program: it is rather a cost-benefit analysis arguing that the "costs of inaction are larger than costs of action." Assuming a path of growth for the GDP, and starting from a Business as Usual (*laissez-faire*) hypothesis on the path of GHG emissions, it considers alternative policies that reduce emissions in the present, and eventually stabilize GHG in the atmosphere. The review argues that, properly discounted, the benefits of strong, early action on climate change outweigh the costs.

It should be noted that discount rates have different roles in Cost-Benefit Analysis and in discounted-utilitarianism optimization. Discounted utilitarianism (see Section A5.4 below) uses the pure time discount rate δ to weight the utilities of the various generations in the utilitarian maximand, whereas Cost-Benefit Analysis uses the consumption discount rate $\delta + \eta \tilde{g}$ to evaluate the changes in future consumption streams due to a particular (marginal) investment project, relative to a reference consumption path that exogenously grows at a rate $\tilde{g}$. The project passes the Cost Benefit test if the discounted sum of the consumption streams is positive. As noted above, the Stern Review uses a pure time discount rate of $\delta = 0.001$ (based on the survival justification), together with $\eta = 1$ and $\tilde{g} = \frac{\dot{c}}{c} = 0.013$ (1.3 % per annum), yielding a *consumption discount rate* of 0.014. Its commentators suggest higher consumption discount rates (Arrow, 2007, Nordhaus, 2007, Martin Weitzman, 2007: see the debate in the *Postscripts to the Stern Review* available at www.sternreview.org.uk, as well as the issue of *World Economics* 7 (4), October-December 2006, and the subsequent Simon Dietz *et al*., 2007). [18]

Because the Stern Review does not solve an optimization program, its recommendations are in principle open to the criticism, voiced by the critics of the Review, that the consumption discount

[18] Nordhaus discounts the utility of future generations by the time-rate of discount that he deduces for today's market consumer, from the Ramsey equation, which he takes to be $\delta = .015$ per annum. This leads to a discount factor applied to the utility of those alive a century from now of $(\frac{1}{1+\delta})^{100} = (\frac{1}{1.015})^{100} = 0.225$. Stern discounts the utility of those a century from now (who may not exist) according to the probability of extinction of the human species; he applies a discount factor of $(1-p)^4 = (.975)^4 = 0.904$. If we adopt Stern's probability-of-extinction, we do not discount the utility (quality of life) of those a century from now at all: that is, our discount factor applied to the utility of those a century from now is unity.

rate should reflect the rates of return of the available investment alternatives: even if, using a consumption discount rate of 0.014, carbon emission reductions pass the Cost-Benefit test, future generations could conceivably be better off if the current generation avoided incurring the costs of GHG reductions and invested instead in other intergenerational public goods. In defense of the Review, Dietz *et al*. (2007, p. 137) argue that "it is hard to know why we should be confident that social rates of return would be, say, 3% or 4% into the future. In particular, if there are strong climate change externalities, then social rates of return on investment may be much lower than the observed private returns on capital over the last century, on which suggestions of a benchmark of 3% or 4% appear to be based."

As we have shown, the discounted utilitarian program with the Stern Review's discount factor diverges on the set of feasible paths that we have proposed in this article. Because the Stern Review only calculates discounted utility for a small number of generations, it need not address this issue. This again shows the limitations of the cost-benefit approach. For further discussion of how an Ethical Observer who is a discounted utilitarian would choose paths when the discounted-utilitarian program diverges, see Llavador *et al.* (2009).

Our approach is in a sense dual to Cost-Benefit analysis. The latter takes as given a path for the economic variables, and recommends a path for the environmental variables (based on a cost-benefit criterion in the spirit of discounted utilitarianism). We, on the contrary, take as given a path for the environmental variables, and recommend paths for the economic variables (based on the criteria of sustainable QuoL and sustainable growth).

7. Summary and conclusions

Our starting point has been a notion of human quality of life, in the spirit of the human development index (HDI), that emphasizes the following three factors in addition to the conventional consumption and leisure.

(i) Education, which modifies the value of leisure time to the individual, in addition to enhancing her productivity;

(ii) Knowledge, in the form of culture and science, which directly improves the living experience, in addition to raising total factor productivity; and

(iii) The quality of the environment.

Because of the importance of climate change, we interpret the environmental variable as the concentration of greenhouse gases (GHG) in the atmosphere. We exogenously specify a path of emissions and associated GHG concentration that climate scientists believe to be physically feasible yielding a stabilized concentration of atmospheric GHG. In line with the consensus expressed in the various IPCC reports and emphasized in the Stern Review, we hypothesize a “catastrophic” level of GHG such that the quality of life tends to zero as the GHG stock approaches this level. We quantify the quality of the environment as the difference between the catastrophic and actual levels.

We adopt social objectives based either on an intergenerational version of the maximin criterion, or on the valuation of sustained growth in the quality of life as a public good. In the first case, the optimization program maximizes the quality of life that can be sustained for all generations. In the second case, we maximize the quality of life of the first generation subject to achieving a given, constant rate of growth for all subsequent generations: this we call the Sustainable Growth Optimization Program. These objectives stand in sharp contrast to the conventional criterion of maximizing the discounted sum of utilities, which we find ethically unjustifiable, at least for the discount factors typically used.

Ideally, for the Maximin Program, we would like to approach paths where all variables are stationary, whereas for the Sustainable Growth Optimization Program we would like to approach balanced-growth paths, where all variables grow at the same rate. But given the current state of climate change science, we cannot confidently adopt a reasonably simple model of emission-stock interaction. In addition, our formulation does not allow the quality of the atmosphere to improve without limit. Accordingly, our computations fix emissions and GHG concentrations at levels that may allow for stabilization after two generations.

The resulting dynamic optimization programs defy explicit analytical solutions, and our approach has been computational. We have devised computational algorithms inspired by the turnpike property for constructing feasible and desirable, although not necessarily optimal, paths in the more complex and interesting models.

In more detail, we have adopted a simplified path for emissions and concentrations that is based on the more elaborate paths proposed in the IPCC 2007 report aiming at stabilizing the concentration of CO_2 in the atmosphere at 450 ppm (in our units, 363 GtC above preindustrial levels). Our simplified version assumes that we jump to a steady state in two generations, after which emissions are maintained at a very low level and the concentration of CO_2 in the atmosphere is stabilized. We have then computed solutions for the economic variables, by an algorithm that mimics the turnpike method.

We conclude that it is possible to sustain quality of life levels higher than the year 2000 reference value, even when maintaining a positive rate of growth for all successive generations. Not surprisingly, higher rates of sustained growth require a lower QuoL for the first generation, but the tradeoff is small, and the first generation reaches a QuoL higher than the reference value for reasonable rates of growth.

Achieving this kind of human sustainability under the postulated environmental path requires particular kinds of behavior for the economic variables. The most important change is doubling the fraction of labor resources devoted to the creation of knowledge, whereas the fractions of labor allocated to consumption and leisure are similar to those of the reference year 2000.

On the other hand, higher growth rates require substantial increases in the fraction of labor devoted to education, together with moderate increases in the fractions of labor devoted to knowledge and the investment in physical capital.

Our analysis departs from the literature in three dimensions: (a) the concept of the quality of life, (b) social welfare criteria, and (c) method. For (a), we adopt a comprehensive notion of the quality of life, which depends not only on consumption and leisure, but also on knowledge, the environment and educated leisure. For (b), we consider both a maximin (or human sustainability) criterion, and a criterion based on the sustainability of quality-of-life growth, where we fix positive rates of growth with the justification than growth has the character of a public good. As for (c), our method is inspired by optimization, but, given the current uncertainties in climate science, we do not attempt to compute an optimal path for environmental variables: we take instead as given a conservative path for the environmental variables, and propose paths for the economic variables based on the criteria of human sustainability and the sustainability of growth in the human quality of life.

APPENDIX 1. BALANCED GROWTH PATHS IN PROGRAM E

Recall that Program $E[\rho, e^*, S^{m*}]$ of Section 3 above assumes that emissions and concentrations are fixed at levels e* and $S^m *$, respectively. It can be written as:

max Λ subject to

$$
\begin{aligned}
&(\lambda_t) \quad c_t^{\alpha_c}(x_t^l)^{\alpha_\ell}(S_t^n)^{\alpha_n}(\hat{S}^m - S^m *)^{\alpha_m} \geq \Lambda(1+\rho)^{t-1}, \text{ for } t \geq 1,\\
&(y_t) \quad k_1(S^{m*})^{\theta_m}(e^*)^{\theta_e}(x_t^c)^{\theta_c}(S_t^k)^{\theta_k}(S_t^n)^{\theta_n} \geq c_t + i_t, t \geq 1,\\
&(w_t) \quad (1-\delta^k)S_{t-1}^k + k_2 i_t \geq S_t^k, t \geq 1,\\
&(n_t) \quad (1-\delta^n)S_{t-1}^n + k_3 x_t^n \geq S_t^n, t \geq 1,\\
&(p_t) \quad k_4 x_{t-1}^e \geq x_t^e + x_t^n + x_t^l + x_t^c, t \geq 1.
\end{aligned}
$$

The Lagrangian multipliers have been written to the left of the constraints. Our problem is to find the condition on the endowment vector (x_0^e, S_0^k, S_0^n) such that the optimal solution to the program is a path of steady growth. At steady-state growth there will be three different growth rates:

- the variables $S_t^n, x_t^n, x_t^e, x_t^c, x_t^l$ will grow at a rate g;
- the variables S_t^k, i_t, c_t will grow at a rate γ;
- Λ_t will grow at a rate ρ.

From the production function, we must have:

$$(1+\gamma) = (1+g)^{\theta_c}(1+\gamma)^{\theta_k}(1+g)^{\theta_n}.$$

However, as we have chosen parameters so that $1-\theta_k = \theta_c + \theta_n$, we have $\gamma = g$, and so there will be only two growth rates, namely g and ρ. From the first constraint, we have $(1+g)^{\alpha_c+\alpha_l+\alpha_n} = 1+\rho$; thus a chosen rate g determines ρ.

Given the ordered triple (g, e^*, S^{m*}), there will be a ray $\Gamma(g, e^*, S^{m*}) \subset \Re_+^3$ such that if the endowment vector $(x_0^e, S_0^n, S_0^k) \in \Gamma(g, e^*, S^{m*})$, then balanced growth at rates g (and ρ) will occur at the optimal solution to the program. We proceed to determine this ray.

To do so, we first derive the Kuhn-Tucker conditions for the program, which are:

$(a)\ (\partial\Lambda) \quad 1-\sum_{t=1}^{\infty}\lambda_t(1+\rho)^{t-1}=0, t\geq 1,$

$(b)\ (\partial x_t^e) \quad k_4 p_{t+1}-p_t=0,$ i. e., $p_t=(1/k_4)^{t-1}p_1, t\geq 1,$

$(c)\ (\partial x_t^l) \quad \frac{\lambda_t\alpha_l\Lambda(1+\rho)^{t-1}}{x_t^l}-p_t=0, t\geq 1,$

$(d)\ (\partial x_t^n) \quad k_3 n_t = p_t, t\geq 1,$

$(e)\ (\partial x_t^c) \quad \frac{y_t\theta_c(c_t+i_t)}{x_t^c}-p_t=0, t\geq 1,$

$(f)\ (\partial c_t) \quad \lambda_t\frac{\alpha_c\Lambda(1+\rho)^{t-1}}{c_t}-y_t=0, t\geq 1,$

$(g)\ (\partial i_t) \quad -y_t+k_2 w_t=0, t\geq 1,$

$(h)\ (\partial S_t^k) \quad \frac{y_t\theta_k(c_t+i_t)}{S_t^k}-w_t+(1-\delta^k)w_{t+1}=0, t\geq 1,$

$(i)\ (\partial S_t^n) \quad \frac{\lambda_t\alpha_n\Lambda(1+\rho)^{t-1}}{S_t^n}+\frac{y_t\theta_n(c_t+i_t)}{S_t^n}+(1-\delta^n)n_{t+1}-n_t=0, t\geq 1.$

We now substitute into these equations the variable values on a balanced growth path.

1. (b) and (c) imply that:

$$\lambda_t=\left(\frac{p_1x_1^l}{\alpha_l\Lambda}\right)\left(\frac{1+g}{k_4(1+\rho)}\right)^{t-1}.$$

2. By (a), it follows that $1=\left[\sum_{t=1}^{\infty}(\frac{1+g}{k_4})^{t-1}\right]\frac{p_1x_1^l}{\alpha_l\Lambda}$. This defines p_1 at the solution, and hence p_t.

Note that p_1 will be defined as long as $k_4>1+g$, so that the series converges. It follows that :

$$1=\left(\frac{p_1x_1^l}{\alpha_l\Lambda}\right)\frac{k_4}{k_4-(1+g)}.$$

3. (d) defines $n_t=p_t/k_3=\frac{p_1}{k_3}(1/k_4)^{t-1}$, whereas (e) defines $y_t\geq 0$, and (g) defines $w_t=y_t/k_2$. Thus all the dual variables are defined and non-negative.

This leaves equations (h), (f) and (i) which we now analyze.

4. Analysis of (h)

(e) implies $y_t = \frac{p_t x_t^c}{\theta_c(c_t + i_t)}$, so (h) says $\frac{p_t x_t^c \theta_k}{\theta_c S_t^k} = \frac{y_t - (1-\delta^k) y_{t+1}}{k_2}$. Substituting for y_t, and multiplying by $\frac{\theta_c}{p_t}$ gives:

$$\frac{k_2 x_t^c \theta_k}{S_t^k} = \frac{x_t^c}{c_t + i_t} - \frac{1-\delta^k}{k_4} \frac{x_{t+1}^c}{c_{t+1} + i_{t+1}},$$

which, using the balanced growth property of the path means:

$$\frac{k_2 x_1^c \theta_k}{S_0^k (1+g)} = \frac{x_1^c}{c_1 + i_1} - \frac{1-\delta^k}{k_4} \frac{x_1^c}{c_1 + i_1}.$$

Multiplying by $\frac{1+g}{x_1^c}$, we have:

(A) $\frac{k_2 \theta_k}{S_0^k} = \frac{(1+g)(1-(\frac{1-\delta^k}{k_4}))}{c_1 + i_1}$.

5. Analysis of (f)

(f) implies $\frac{\lambda_t \alpha_c \Lambda (1+\rho)^{t-1}}{c_t} = \frac{p_t x_t^c}{\theta_c (c_t + i_t)}$ which may be reduced to the equation:

(B) $\frac{x_1^l}{\alpha_l} = \frac{c_1 x_1^c}{\alpha_c \theta_c (c_1 + i_1)}$.

6. Analysis of (i)

We express λ_t, y_t, n_t in terms of p_t; after some algebraic manipulation (i) reduces to:

(C) $\frac{\alpha_n x_1^l}{\alpha_l (1+g) S_0^n} + \frac{\theta_n x_1^c}{\theta_c (1+g) S_0^n} + \frac{1}{k_3}\left(\frac{1-\delta^n}{k_4} - 1\right) = 0$.

In sum, we have the three equations (A), (B), and (C). From the primal constraints we have:

(D) $k_1 (1+g)^{\theta_n + \theta_k} (S_0^k)^{\theta_k} (S_0^n)^{\theta_n} (x_1^c)^{\theta_c} (S^m *)^{\theta_m} (e*)^{\theta_e} = c_1 + i_1$,

(E) $x_0^e (k_4 - (1+g)) = x_1^n + x_1^c + x_1^l$,

(F) $k_2 i_1 = S_0^k (g + \delta^k)$,

(G) $k_3 x_1^n = (g + \delta^n) S_0^n$.

Define the following expressions:

$$v^n(g) = \frac{\delta^n + g}{k_3},$$

$$p^i(g) = \frac{\delta^k + g}{k_2},$$

$$p^c(g) = \left(1 - \frac{1-\delta^k}{k_4}\right)\frac{1+g}{k_2\,\theta_k} - \frac{\delta^k + g}{k_2},$$

$$v^l(g) = \left(1 - \frac{1-\delta^n}{k_4}\right)\frac{\alpha_l}{k_3}\frac{p^c(g)}{\alpha_c\theta_n\left(p^c(g)+p^i(g)\right)+\alpha_n p^c(g)}\left(1+g\right),$$

$$v^c(g) = \left(1 - \frac{1-\delta^n}{k_4}\right)\frac{\alpha_c\,\theta_c}{k_3}\frac{p^c(g)+p^i(g)}{\alpha_c\theta_n\left(p^c(g)+p^i(g)\right)+\alpha_n p^c(g)}\left(1+g\right),$$

$$q^n(g) = \frac{k_4 - \left(1+g\right)}{v^n(g)+v^l(g)+v^c(g)},$$

and $q^k\left(g, e^*, S^{m*}\right) = \left(\frac{k_1\left(v^c(g)\right)^{\theta_c}}{p^c(g)+p^i(g)}\right)^{\frac{1}{\theta_n+\theta_c}}(1+g)^{\frac{\theta_n+\theta_k}{\theta_n+\theta_c}}\cdot q^n(g)\cdot(e*)^{\frac{\theta_e}{\theta_n+\theta_c}}\cdot\left(S^{m*}\right)^{\frac{\theta_m}{\theta_n+\theta_c}}$.

Note that these seven functions are all positive. In particular, it is easily checked that $p^c(g) > 0$ for any $g \geq 0$, since $k_4 > 1$.

Now from (F) we solve for *i*:

$$i_1 = p^i(g)S_0^k.$$

From (G), we have

$$x_1^n = v^n(g)S_0^n.$$

From (A) and the above expression for i_1, we have:

$$c_1 = p^c(g)S_0^k.$$

Now view (B) and (C) as a pair of simultaneous linear equations in (x_1^c, x_1^l). Solving them gives

$$(x_1^c, x_1^l) = (\nu^c(g), \nu^l(g))S_0^n.$$

Substituting these values into (E) gives

$$S_0^n = q^n(g)x_0^e.$$

Finally, we obtain

$$S_0^k = q^k(g, e^*, S^{m*})x_0^e$$

by substituting $S_0^n = q^n(g)x_0^e$ and $x^c = \nu^c(g)q^n(g)x_0^e$ into equation (D) and solving for S_0^k.

Statement (ii) of Theorem 1 is immediately derived from the above equations. Statement (i) asserts that the endowments grow along the ray $\Gamma(g, e^*, S^{m*})$ at rate 1 + *g*, and statement (iii) says that all flow variables exhibit balanced growth. ■

APPENDIX 2. REACHING THE RAY $\Gamma(g, e^*, S^{m*})$ IN TWO GENERATIONS FROM DATE-2000 ENDOWMENTS

The ray $\Gamma(g, e^*, S^{m*})$ is defined by

$$\Gamma(g, e^*, S^{m*}) = \{(x^e, S^k, S^n) \in \Re_+^3 : S^k = q^k(g, e^*, S^{m*})x^e, S^n = q^n(g)x^e\},$$

where the coefficients q^n and q^k have been computed in Appendix 1 above. Program $G[x_2^e]$ of Section 3 above can now be written as follows.

<u>Program</u> $G[x_2^e]$: *G*iven $(\rho, e_1, S_1^m, e_2, S_2^m, e^*, S^{m*})$ and x_2^e, Max Λ_1 subject to

(A3.1) $(\mu_1) : (c_1)^{\alpha_c}(x_1^l)^{\alpha_l}(S_1^n)^{\alpha_n}(\hat{S}^m - S_1^m)^{\alpha_m} \geq \Lambda_1,$

(A3.2) $(\mu_2) : (c_2)^{\alpha_c}(x_2^l)^{\alpha_l}(q^n(g)x_2^e)^{\alpha_n}(\hat{S}^m - S_2^m)^{\alpha_m} \geq (1+\rho)\Lambda_1,$

(A3.3) $(r_1) : k_1(x_1^c)^{\theta_c}(S_1^k)^{\theta_k}(S_1^n)^{\theta_n}(e_1)^{\theta_e}(S_1^m)^{\theta_m} \geq c_1 + i_1,$

(A3.4) $(r_2) : k_1(x_2^c)^{\theta_c}(q^k(g, e^*, S^{m*})x_2^e)^{\theta_k}(q^n(g)x_2^e)^{\theta_n}(e_2)^{\theta_e}(S_2^m)^{\theta_m} \geq c_2 + i_2,$

(A3.5) $(z_1) : (1-\delta^k)S_0^k + k_2 i_1 \geq S_1^k,$

(A3.6) $(z_2) : (1-\delta^k)S_1^k + k_2 i_2 \geq q^k(g, e^*, S^{m*})x_2^e,$

(A3.7) $(\beta_1) : (1-\delta^n)S_0^n + k_3 x_1^n \geq S_1^n,$

(A3.8) $(\beta_2) : (1-\delta^n)S_1^n + k_3 x_2^n \geq q^n(g)x_2^e,$

(A3.9) $(\zeta_1) : k_4 x_0^e \geq x_1^e + x_1^n + x_1^l + x_1^c,$

(A3.10) $(\zeta_2) : k_4 x_1^e \geq x_2^e + x_2^n + x_2^l + x_2^c,$

for the year-2000 initial conditions $(x_0^e, S_0^k, S_0^n) = (\bar{x}_{2000}^e, \bar{S}_{2000}^k, \bar{S}_{2000}^n)$.

This is a concave program, and therefore the first-order conditions will be sufficient. We have 10 constraints and hence 10 Lagrangian multipliers, shown to the left of each constraint. There are 14 endogenous variables $(\Lambda_1, c_1, x_1^l, x_1^c, x_1^n, x_1^e, c_2, x_2^l, x_2^c, x_2^n, i_1, i_2, S_1^k, S_1^n)$, and hence 14 Kuhn-Tucker conditions, as follows.

$$KT1:(\partial\Lambda_1) \quad 1-\mu_1-(1+\rho)\mu_2=0;$$

$$KT2:(\partial c_1) \quad \mu_1\frac{\alpha_c\Lambda_1}{c_1}-r_1=0;$$

$$KT3:(\partial c_2) \quad \mu_2\frac{\alpha_c(1+\rho)\Lambda_1}{c_2}-r_2=0;$$

$$KT4:(\partial x_1^l) \quad \mu_1\frac{\alpha_l\Lambda_1}{x_1^l}-\zeta_1=0;$$

$$KT5:(\partial x_2^l) \quad \mu_2\frac{\alpha_l(1+\rho)\Lambda_1}{x_2^l}-\zeta_2=0;$$

$$KT6:(\partial S_1^k) \quad r_1\frac{\theta_k(c_1+i_1)}{S_1^k}-z_1+z_2(1-\delta^k)=0;$$

$$KT7:(\partial S_1^n) \quad \mu_1\frac{\alpha_n\Lambda_1}{S_1^n}+r_1\frac{\theta_n(c_1+i_1)}{S_1^n}-\beta_1+(1-\delta^n)\beta_2=0;$$

$$KT8:(\partial x_1^e) \quad -\zeta_1+\zeta_2 k_4=0;$$

$$KT9:(\partial x_1^n) \quad \beta_1 k_3-\zeta_1=0;$$

$$KT10:(\partial x_2^n) \quad \beta_2 k_3-\zeta_2=0;$$

$$KT11:(\partial x_1^c) \quad r_1\frac{\theta_c(c_1+i_1)}{x_1^c}-\zeta_1=0;$$

$$KT12:(\partial x_2^c) \quad r_2\frac{\theta_c(c_2+i_2)}{x_2^c}-\zeta_2=0;$$

$$KT13:(\partial i_1) \quad -r_1+z_1 k_2=0;$$

$$KT14:(\partial i_2) \quad -r_2+z_2 k_2=0.$$

(a) From KT11, $\frac{\zeta_1}{r_1}=\theta_c\frac{c_1+i_1}{x_1^c}$. From KT4 and KT2, $\frac{\zeta_1}{r_1}=\frac{\mu_1\alpha_l\Lambda_1}{x_1^l}\frac{1}{\mu_1\alpha_c\Lambda_1}c_1=\frac{\alpha_l c_1}{\alpha_c x_1^l}$. It follows that

$$c_1=\frac{\theta_c\alpha_c x_1^l}{\alpha_l x_1^c-\theta_c\alpha_c x_1^l}i_1. \qquad \text{(a.1)}$$

Similarly, from KT12, $\frac{\zeta_2}{r_2} = \theta_c \frac{c_2 + i_2}{x_2^c}$. From KT5 and KT3,

$\frac{\zeta_2}{r_2} = \frac{\mu_2 \alpha_l (1+\rho)\Lambda_1}{x_2^l} \frac{1}{\mu_2 \alpha_c (1+\rho)\Lambda_1} c_2 = \frac{\alpha_l c_2}{\alpha_c x_2^l}$, yielding

$$c_2 = \frac{\theta_c \alpha_c x_2^l}{\alpha_l x_2^c - \theta_c x_2^l \alpha_c} i_2. \qquad \text{(a.2)}$$

(b) From KT8

$$\frac{\zeta_2}{\zeta_1} = \frac{1}{k_4}, \qquad \text{(b.1)}$$

whereas from KT9 and KT10,

$$\frac{\beta_2}{\beta_1} = \frac{\zeta_2}{\zeta_1}, \qquad \text{(b.2)}$$

yielding

$$\frac{\beta_2}{\beta_1} = \frac{1}{k_4}. \qquad \text{(b.3)}$$

From KT4 and KT9

$$\mu_1 \frac{\alpha_l \Lambda_1}{x_1^l} = \beta_1 k_3, \qquad \text{(b.4)}$$

and from KT5 and KT10

$$\mu_2 \frac{\alpha_l (1+\rho)\Lambda_1}{x_2^l} = \beta_2 k_3. \qquad \text{(b.5)}$$

Dividing (b.5) by (b.4)

$$\frac{\mu_2 x_1^l}{\mu_1 x_2^l}(1+\rho) = \frac{\beta_2}{\beta_1}, \qquad \text{(b.6)}$$

which together with (b.3) yields

$$k_4 \mu_2 \frac{x_1^l}{x_2^l}(1+\rho) = \mu_1. \qquad \text{(b.7)}$$

Substituting (b.7) into KT1 gives

$$1 = k_4 \mu_2 \frac{x_1^l}{x_2^l}(1+\rho) + (1+\rho)\mu_2,$$

i. e., $$\mu_2(1+\rho)\left[k_4 \frac{x_1^l}{x_2^l}+1\right]=1,$$

or: $$\mu_2 = \frac{x_2^l}{(1+\rho)(k_4 x_1^l + x_2^l)}, \tag{b.8}$$

which together with (b.7) yields

$$\mu_1 = \frac{k_4 x_1^l}{k_4 x_1^l + x_2^l}. \tag{b.9}$$

From (b.4) and (b.9),

$$\beta_1 = \frac{\mu_1}{k_3}\frac{\alpha_l \Lambda_1}{x_1^l} = \frac{k_4 \alpha_l x_1^l \Lambda_1}{k_3(k_4 x_1^l + x_2^l)x_1^l},$$

i. e., $$\beta_1 = \frac{k_4 \alpha_l \Lambda_1}{k_3(k_4 x_1^l + x_2^l)}, \tag{b.10}$$

and from (b.5) and (b.8),

$$\beta_2 = \frac{\mu_2}{k_3}\frac{\alpha_l(1+\rho)\Lambda_1}{x_2^l} = \frac{\alpha_l x_2^l (1+\rho)\Lambda_1}{k_3(k_4 x_1^l + x_2^l)(1+\rho)x_2^l},$$

i. e., $$\beta_2 = \frac{\alpha_l \Lambda_1}{k_3(k_4 x_1^l + x_2^l)}. \tag{b.11}$$

From KT9, $\zeta_1 = \beta_1 k_3$, i. e., using (b.10),

$$\zeta_1 = \frac{k_4 \alpha_l \Lambda_1}{k_4 x_1^l + x_2^l}, \tag{b.12}$$

and, similarly, from KT10 and (b.11).

$$\zeta_2 = \frac{\alpha_l \Lambda_1}{k_4 x_1^l + x_2^l}. \tag{b.13}$$

Finally, from KT2 and (b.9),

$$r_1 = \frac{k_4 \alpha_c x_1^l \Lambda_1}{(k_4 x_1^l + x_2^l)c_1}, \tag{b.14}$$

and from KT3 and (b.8)

$$r_2 = \frac{x_2^l}{(1+\rho)(k_4 x_1^l + x_2^l)}\frac{\alpha_c(1+\rho)\Lambda_1}{c_2},$$

i.e., $$r_2 = \frac{\alpha_c x_2^l \Lambda_1}{(k_4 x_1^l + x_2^l)c_2}. \tag{b.15}$$

From KT13 and (b.14)

$$z_1 = \frac{r_1}{k_2} = \frac{k_4 \alpha_c x_1^l \Lambda_1}{k_2 (k_4 x_1^l + x_2^l) c_1}, \tag{b.16}$$

and from KT14 and (b.15)

$$z_2 = \frac{r_2}{k_2} = \frac{x_2^l \alpha_c \Lambda_1}{k_2 (k_4 x_1^l + x_2^l) c_2}. \tag{b.17}$$

(c) Inserting (b.14), (b.9), (b.10) and (b.11) into KT7:

$$\mu_1 \frac{\alpha_n \Lambda_1}{S_1^n} + r_1 \frac{\theta_n (c_1 + i_1)}{S_1^n} - \beta_1 + (1-\delta^n)\beta_2 = 0,$$

we obtain $\frac{k_4 x_1^l}{k_4 x_1^l + x_2^l} \frac{\alpha_n \Lambda_1}{S_1^n} + \frac{k_4 x_1^l \alpha_c \Lambda_1}{(k_4 x_1^l + x_2^l) c_1} \frac{\theta_n (c_1 + i_1)}{S_1^n} - \frac{k_4 \alpha_l \Lambda_1}{k_3 (k_4 x_1^l + x_2^l)} + (1-\delta^n) \frac{\alpha_l \Lambda_1}{k_3 (k_4 x_1^l + x_2^l)} = 0,$

i. e., $$\frac{k_4 \alpha_n x_1^l}{S_1^n} + \frac{k_4 \alpha_c x_1^l \theta_n (c_1 + i_1)}{c_1 S_1^n} - \frac{k_4 \alpha_l}{k_3} + (1-\delta^n) \frac{\alpha_l}{k_3} = 0. \tag{c.1}$$

Inserting (b.14), (b.16), and (b.17) into KT6:

$$r_1 \frac{\theta_k (c_1 + i_1)}{S_1^k} - z_1 + z_2 (1-\delta^k) = 0,$$

we obtain $\frac{k_4 x_1^l \alpha_c \Lambda_1}{(k_4 x_1^l + x_2^l) c_1} \frac{\theta_k (c_1 + i_1)}{S_1^k} - \frac{k_4 x_1^l \alpha_c \Lambda_1}{k_2 (k_4 x_1^l + x_2^l) c_1} + \frac{x_2^l \alpha_c \Lambda_1}{k_2 (k_4 x_1^l + x_2^l) c_2} (1-\delta^k) = 0,$

i. e., $$\frac{k_4 x_1^l \theta_k (c_1 + i_1)}{c_1 S_1^k} - \frac{k_4 x_1^l}{k_2 c_1} + \frac{x_2^l (1-\delta^k)}{k_2 c_2} = 0. \tag{c.2}$$

(d) In summary, the Kuhn-Tucker conditions yield the following four equations involving only primal variables, which added to the 10 constraints, written as equalities, constitute a system of 14 equations in the 14 primal variables. The four equations are:

$$c_1 = \frac{\theta_c \alpha_c x_1^l}{\alpha_l x_1^c - \theta_c \alpha_c x_1^l} i_1, \tag{a.1}$$

$$c_2 = \frac{\theta_c \alpha_c x_2^l}{\alpha_l x_2^c - \theta_c \alpha_c x_2^l} i_2, \tag{a.2}$$

$$\frac{k_4 \alpha_n x_1^l}{S_1^n} + \frac{k_4 \alpha_c x_1^l \theta_n (c_1 + i_1)}{c_1 S_1^n} - \frac{k_4 \alpha_l}{k_3} + (1-\delta^n) \frac{\alpha_l}{k_3} = 0, \tag{c.1}$$

and $$\frac{k_4 x_1^l \theta_k (c_1 + i_1)}{c_1 S_1^k} - \frac{k_4 x_1^l}{k_2 c_1} + \frac{x_2^l (1-\delta^k)}{k_2 c_2} = 0 \,. \qquad \text{(c.2)}$$

(e) From (A3.5), $$i_1 = \frac{S_1^k - (1-\delta^k) S_0^k}{k_2}, \qquad \text{(e.1)}$$

which substituted into (a.1) yields $$c_1 = \frac{\theta_c \alpha_c x_1^l}{\alpha_l x_1^c - \theta_c \alpha_c x_1^l} \frac{S_1^k - (1-\delta^k) S_0^k}{k_2} \qquad \text{(e.2)}$$

and $c_1 + i_1 = \left[\frac{\theta_c \alpha_c x_1^l}{\alpha_l x_1^c - \theta_c \alpha_c x_1^l} + 1 \right] \frac{S_1^k - (1-\delta^k) S_0^k}{k_2}$, i. e.,

$$c_1 + i_1 = \alpha_l x_1^c \frac{S_1^k - (1-\delta^k) S_0^k}{k_2 (\alpha_l x_1^c - \theta_c \alpha_c x_1^l)}, \qquad \text{(e.3)}$$

which in turn gives

$$\frac{c_1 + i_1}{c_1} = \frac{\alpha_l x_1^c}{\theta_c \alpha_c x_1^l} \,. \qquad \text{(e.4)}$$

Similarly, from (A3.6), $$i_2 = \frac{q^k x_2^e - (1-\delta^k) S_1^k}{k_2}, \qquad \text{(e.5)}$$

which substituted into (a.2) yields

$$c_2 = \frac{\theta_c \alpha_c x_2^l}{\alpha_l x_2^c - \theta_c \alpha_c x_2^l} \frac{q^k x_2^e - (1-\delta^k) S_1^k}{k_2} \qquad \text{(e.6)}$$

and

$$c_2 + i_2 = \alpha_l x_2^c \frac{q^k x_2^e - (1-\delta^k) S_1^k}{k_2 (\alpha_l x_2^c - \theta_c \alpha_c x_2^l)}, \qquad \text{(e.7)}$$

which in turn gives

$$\frac{c_2 + i_2}{c_2} = \frac{\alpha_l x_2^c}{\theta_c \alpha_c x_2^l} \,. \qquad \text{(e.8)}$$

From (A3.7)

$$x_1^n = \frac{S_1^n - (1-\delta^n) S_0^n}{k_3}, \qquad \text{(e.9)}$$

and from (A3.8)

$$x_2^n = \frac{q^n x_2^e - (1-\delta^n) S_1^n}{k_3} \,. \qquad \text{(e.10)}$$

(f) Inserting (e.3) into (A3.3) we obtain

$$k_1(x_1^c)^{\theta_c}(S_1^k)^{\theta_k}(S_1^n)^{\theta_n}(e_1)^{\theta_e}(S_1^m)^{\theta_m}-\alpha_l x_1^c\frac{S_1^k-(1-\delta^k)S_0^k}{k_2(\alpha_l x_1^c-\theta_c\alpha_c x_1^l)}=0, \qquad \text{(f.1)}$$

an equation of the form $\varphi_1(x_1^c,x_1^l,S_1^k,S_1^n)=0$, while inserting (e.7) into (A3.4) yields

$$k_1(x_2^c)^{\theta_c}(q^k x_2^e)^{\theta_k}(q^n x_2^e)^{\theta_n}(e_2)^{\theta_e}(S_2^m)^{\theta_m}-\alpha_l x_2^c\frac{q^k x_2^e-(1-\delta^k)S_1^k}{k_2(\alpha_l x_2^c-\theta_c\alpha_c x_2^l)}=0, \qquad \text{(f.2)}$$

an equation of the form $\varphi_2(x_2^c,x_2^l,S_1^k)=0.$

Inserting (e.4) into (c.1), we obtain

$$\frac{k_4x_1^l\alpha_n}{S_1^n}+\frac{k_4x_1^l\alpha_c\theta_n}{S_1^n}\frac{\alpha_l x_1^c}{\theta_c\alpha_c x_1^l}-\frac{k_4\alpha_l}{k_3}+(1-\delta^n)\frac{\alpha_l}{k_3}=0,$$

or: $\theta_c k_3k_4\alpha_n x_1^l+k_3k_4\theta_n\alpha_l x_1^c+\theta_c\alpha_l[1-\delta^n-k_4]S_1^n=0,$ (f.3)

a linear equation of the form $\varphi_3(x_1^c,x_1^l,S_1^n)=0.$

Inserting (e.4), (e.2) and (e.6) into (c.2) yields

$$\frac{k_4x_1^l\theta_k}{S_1^k}\frac{\alpha_l x_1^c}{\theta_c\alpha_c x_1^l}-\frac{k_4x_1^l}{k_2\dfrac{\theta_c\alpha_c x_1^l}{\alpha_l x_1^c-\theta_c\alpha_c x_1^l}\dfrac{S_1^k-(1-\delta^k)S_0^k}{k_2}}+\frac{x_2^l(1-\delta^k)}{k_2\dfrac{\theta_c\alpha_c x_2^l}{\alpha_l x_2^c-\theta_c\alpha_c x_2^l}\dfrac{q^k x_2^e-(1-\delta^k)S_1^k}{k_2}}=0,$$

i. e., $\dfrac{k_4\theta_k\alpha_l x_1^c}{S_1^k\theta_c\alpha_c}-\dfrac{k_4(\alpha_l x_1^c-\theta_c\alpha_c x_1^l)}{\theta_c\alpha_c(S_1^k-(1-\delta^k)S_0^k)}+\dfrac{(1-\delta^k)(\alpha_l x_2^c-\theta_c\alpha_c x_2^l)}{\theta_c\alpha_c(q^k x_2^e-(1-\delta^k)S_1^k)}=0,$ (f.4)

an equation of the form $\varphi_4(x_1^c,x_1^l,x_2^c,x_2^l,S_1^k)=0.$

Inserting (e.9) into (A3.9), we obtain

$$x_1^e+\frac{S_1^n-(1-\delta^n)S_0^n}{k_3}+x_1^l+x_1^c-k_4x_0^e=0 \quad , \qquad \text{(f.5)}$$

a linear equation of the form $\varphi_5(x_1^c,x_1^l,x_1^e,S_1^n)=0$, whereas the insertion of (e.10) into (A3.10) yields

$$x_2^e+\frac{q^n x_2^e-(1-\delta^n)S_1^n}{k_3}+x_2^l+x_2^c-k_4x_1^e=0 \quad , \qquad \text{(f.6)}$$

a linear equation of the form $\varphi_6(x_2^c,x_2^l,x_1^e,S_1^n)=0$.

Finally, from (A3.1) and (A3.2), we have

$$(c_2)^{\alpha_c}(x_2^l)^{\alpha_l}(q^n x_2^e)^{\alpha_n}(\hat{S}^m-S_2^m)^{\alpha_m}=(1+\rho)(c_1)^{\alpha_c}(x_1^l)^{\alpha_l}(S_1^n)^{\alpha_n}(\hat{S}^m-S_1^m)^{\alpha_m}.$$

Inserting (e.6) and (e.2) into this equation yields

$$\left(\frac{x_2^l[q^k x_2^e-(1-\delta^k)S_1^k]}{\alpha_l x_2^c-\theta_c\alpha_c x_2^l}\right)^{\alpha_c}(x_2^l)^{\alpha_l}(q^n x_2^e)^{\alpha_n}(\hat{S}^m-S_2^m)^{\alpha_m}$$

$$=(1+\rho)\left(\frac{x_1^l\left[S_1^k-(1-\delta^k)S_0^k\right]}{\alpha_l x_1^c-\theta_c\alpha_c x_1^l}\right)^{\alpha_c}(x_1^l)^{\alpha_l}(S_1^n)^{\alpha_n}(\hat{S}^m-S_1^m)^{\alpha_m}, \quad \text{(f.7)}$$

an equation of the form $\varphi_7(x_1^c, x_1^l, x_2^c, x_2^l, S_1^k, S_1^n)=0$.

The seven equations (f.1) to (f.7) form a system in the seven unknowns $(x_1^c, x_1^l, x_1^e, x_2^c, x_2^l, S_1^k, S_1^n)$. We numerically solve these seven equations using *Mathematica*, and then compute all the other values (including Λ_1, which can be obtained from (A3.1)). We check that all values and Lagrangian multipliers are non-negative to be assured that we have found a solution.

APPENDIX 3. CALIBRATIONS

We interpret that generations live for 25 years. In this appendix, flow variables are typically defined as per year averages, and it is understood that stocks are located in the last year of life of a generation. The calibrated values that we obtain are reported in Section 2.6 above.

A3.1. Variables

S_t^k = capital stock available to Generation *t* (in thousands of dollars per capita).

S_t^n = stock of knowledge available to Generation *t* (in thousands of dollars per capita).

S_t^m = CO_2 concentration in the atmosphere above the equilibrium pre-industrial level at the end of Generation *t*'s life (in GtC).

x_t = average annual efficiency units of time (labor and leisure) available to Generation *t* (in efficiency units per capita).

x_t^e = average annual labor devoted to education by Generation *t* (in efficiency units per capita).

x_t^c = average annual labor devoted to the production of output by Generation *t* (efficiency units per capita).

x_t^l = annual average leisure by Generation *t* (in efficiency units per capita).

x_t^n = average annual labor devoted to the production of knowledge by Generation *t* (in efficiency units per capita).

c_t = annual average consumption by Generation t (in thousands of dollars per capita).

i_t = average annual investment by Generation t (in thousands of dollars per capita).

e_t = average annual emissions of CO_2 from fuel and cement in GtC by Generation t (in GtC).

A3.2. Parameters

α_j = exponents of the QuoL function for $j \in \{c$ (consumption), l (leisure), n (stock of knowledge), and m (quality of the biosphere)$\}$.

k_1 = parameter of the production function f.

k_2 = parameter of the law of motion of capital.

k_3 = parameter of the law of motion of the stock of knowledge.

k_4 = parameter of the education production function.

θ_j = exponents of the inputs in the production function f for $j \in \{c$ (labor), k (stock of capital), n (stock of knowledge), e (emissions of CO_2), m (atmospheric carbon concentration)$\}$.

δ^k = depreciation rate of the stock of capital (per generation).

δ^n = depreciation rate of the stock of knowledge (per generation).

$\hat{S}^m$ = catastrophic level of carbon concentration in the atmosphere above the equilibrium pre-industrial level (in GtC).

ρ = generational rate of growth of the QuoL.

$\hat{\rho}$ = annual rate of growth of the QuoL ($\rho = (1+\hat{\rho})^{25}$).

A3.3. Functions

Quality-of-life function (QuoL): $\tilde{\Lambda}(c_t, x_t^l, S_t^n, S_t^m) \equiv (c_t)^{\alpha_c} (x_t^l)^{\alpha_l} (S_t^n)^{\alpha_n} (\hat{S}^m - S_t^m)^{\alpha_m}$.

Production function: $f(x_t^c, S_t^k, S_t^n, e_t, S_t^m) \equiv k_1 (x_t^c)^{\theta_c} (S_t^k)^{\theta_k} (S_t^n)^{\theta_n} (e_t)^{\theta_e} (S_t^m)^{\theta_m}$, $\theta_c + \theta_c + \theta_n = 1$.

Law of motion of physical capital: $S_t^k \leq (1-\delta^k) S_{t-1}^k + k_2\, i_t$.

Law of motion of the stock of knowledge: $S_t^n \leq (1-\delta^n) S_{t-1}^n + k_3\, x_t^n$.

Education production function: $x_t \leq k_4 x_{t-1}^e$.

A3.4. The calibration of the Quality-of-Life (QuoL) function

We take the exponent of leisure to be twice that of consumption ($\alpha_l = 2\ \alpha_c$) and calibrate $\alpha_n/\alpha_c = 0.05$ as the average ratio of expenditure in knowledge (R&D expenditure plus investment in computer components and software) over expenditure in consumption during the period 1953-2000.[19]

Next, we calibrate the ratio α_m/α_c by the Stern Review (2007) statement that a 5ºC increase in the global temperature over the pre-industrial level would imply a health related damage equivalent to a 5% loss of global GDP (page x).[20] We can read the statement of the Stern Review as saying that a 5% decrease in consumption is equivalent to suffering an atmospheric CO_2 concentration of $\tilde{S}^m$, yielding

$$(.95c)^{\alpha_c}(x^l)^{\alpha_l}(S^n)^{\alpha_n}(\hat{S}^m - S^m)^{\alpha_m} = (c)^{\alpha_c}(x^l)^{\alpha_l}(S^n)^{\alpha_n}(\hat{S}^m - \tilde{S}^m)^{\alpha_m},$$

that is,

$$(.95)^{\alpha_c}(\hat{S}^m - S^m)^{\alpha_m} = (\hat{S}^m - \tilde{S}^m)^{\alpha_m}.$$

Taking logs,

$$\alpha_c \ln(0.95) = \alpha_m \left(\ln\left(\hat{S}^m - \tilde{S}^m\right) - \ln\left(\hat{S}^m - S^m\right)\right).$$

We take a 5ºC increase in temperature to be associated with CO_2 equivalent (CO_2e) concentrations of 1470 GtC (Stern 2007, Figure 2 in page v). Because we only consider CO_2 emissions (which account for 84% of all GHG) and we compute values above pre-industrial level (595.5 GtC), we adopt the value $\tilde{S}^m = \frac{1470}{1.16} - 595.5 = 671.64$ GtC.

We consider that an increase in temperature of 6º-8ºC (relative to pre-industrial level) would have catastrophic impacts.[21] We take this temperature increases to be associated with CO_2 equivalent

[19] The data on R&D are derived from Research and Development in Industry, Academic Research and Development Expenditures, Federal Funds for Research and Development, and the Survey of Research and Development Funding and Performance by Nonprofit Organizations (National Science Foundation, 2003). Data on public investment in software are constructed taking the value of public investment in equipment and software (U.S. Bureau of Economic Analysis 2007) and assuming the same share of software in private and public investment.

[20] This is also in line with Nordhaus and Boyer (2000) who estimate a total cost (market and non-market) of between 9% and 11% of global GDP for a 6ºC warming (as quoted in Stern, 2007, p. 148).

[21] The Stern Review consistently associates catastrophic consequences to temperature increases of 6-8ºC, like, for example, sea level rise threatening major world cities (including London, Shanghai, New York, Tokyo and Hong Kong), entire regions experiencing major declines in crop yields and high risk of abrupt, large scale shifts in the

concentrations of 750 ppm (or 1597.5 GtC), the lower bound of the studies reported in the Stern Review (2007, p.12). As before, adjusting for all gases and subtracting pre-industrial levels, we obtain $\hat{S}^m = \frac{1597.5}{1.16} - 595.5 = 781.55$.

It follows that

$$\frac{\alpha_m}{\alpha_c} = \frac{\ln 0.95}{\ln(781.55 - 671.64) - \ln(781.55 - 177.1)} = 0.03.$$[22]

Finally, we normalize $\alpha_c + \alpha_l + \alpha_m + \alpha_n = 1$, and obtain the values reported in Table 1 of the main text.

A3.5. The calibration of the production function

We construct time series for the stocks of physical capital, knowledge, and human capital, see sections A3.6-8 below. We take the labor income share to be 2/3, and compute the average share of physical capital and knowledge in the total stock of capital for the period 1960-2000, corresponding to 5/6 and 1/6, respectively. We calibrate with these data the production function

$$f(x_t^c, S_t^k, S_t^n, e_t, S_t^m) \equiv k_1 (x_t^c)^{\theta_c} (S_t^k)^{\theta_k} (S_t^n)^{\theta_n} (e_t)^{\theta_e} (S_t^m)^{\theta_m},$$

in the following inputs: first the more usual labor, physical capital and knowledge, to which we add the environmental stock and emissions. We assume constant returns to scale in the first three inputs, i. e., $\theta_c + \theta_k + \theta_n = 1$. Hence, $\theta_c = 2/3$, $\theta_k = 5/18$ and $\theta_n = 1/18$, representing the income share of each input.

We calibrate $\theta_e = 0.091$ as the "elasticity of output with respect to carbon services" from RICE99 in Nordhaus and Boyer (2000).

For the calibration of θ_m, the elasticity of output to the CO_2 concentration in the atmosphere, we assume that doubling the CO_2 concentration from pre-industrial levels would increase temperature by 2.5ºC (Stern, 2007, p.7),[23] and that a 2.5ºC increase in temperature is associated with a 1.5% loss of total GDP (Nordhaus and Boyer, 2000, p.91). Hence,

climate system (Figure 2 in page v), and catastrophic major disruptions and large-scale movements of population (Table 3.1 in p. 57).

[22] As a reference, the US currently devotes approximately 2% of its gross domestic product to all forms of environmental protection.

[23] The Stern Review asserts that temperature would increase 1.5º-4.5ºC (if we consider feedback effects) and 1ºC as direct effects.

$$\theta_m = \frac{\%\Delta\, y}{\%\Delta\, S^m} = \frac{\%\Delta\, y}{\%\Delta T}\frac{\%\Delta T}{\%\Delta\, S^m} = -\frac{.015}{2} = -.0075,$$

where y is GDP per capita and T is global temperature.

We compute k_1 as the TFP of the US economy calibrated to year 2000 values:[24]

$$k_1 = \frac{y_{2000}^{USA}}{\left(x_{2000}^{c,USA}\right)^{\theta_c}\left(S_{2000}^{k,USA}\right)^{\theta_k}\left(S_{2000}^{n,USA}\right)^{\theta_n}\left(e_{2000}^{USA}\right)^{\theta_e}\left(S_{2000}^{m}\right)^{\theta_m}}$$

$$= \frac{34.78}{0.3955^{0.67}\, 73.65^{0.28}\, 15.64^{0.06}\, 1.6^{0.091}\, 177.1^{-0.0075}} = 16.408$$

A3.6. The calibration of the law of motion of the stock of physical capital

Physical capital investment is equal to private plus public investment less investment in software. We take $\hat{\delta}^k = 0.06$ as the annual rate of depreciation (Thomas Cooley and Edward Prescott, 1995). In generational terms, $\delta^k = 0.787$.

To approximate the year-to-year discounting, we take i = average investment in physical capital of Generation t per year, and compute that, at the end of Generation t's life, the accumulated investment amounts are

$$i + i\times(1-\hat{\delta}^k) + i\times(1-\hat{\delta}^k)^2 + \cdots + i\times(1-\hat{\delta}^k)^{24} = \frac{1-(1-\hat{\delta}^k)^{25}}{1-(1-\hat{\delta}^k)}\, i\ .$$

Thus, since $1-\hat{\delta}^k = 0.94$, the parameter $k_2 = \frac{1-(1-\hat{\delta}^k)^{25}}{1-(1-\hat{\delta}^k)} = 13.118$.

The time series of the stock of capital is constructed by the perpetual inventory method, using US data for 1960-2000 and taking 1960 as initial value. For the initial year,

$S_{1960}^k = \frac{i_{1960}^k}{\hat{\delta}^k + g^k} = \frac{2.51}{0.06+0.038} = 25.63$ thousands of constant 2000 dollars per capita, where i^k represents total (private and public) investment per capita minus expenditure in software, and g^k represents the average yearly growth rate of investment between 1960-1970 (set at 0.038). The value for the stock of physical capital in the year 2000 is $\bar{S}_{2000}^k = 73.65$ (in thousands of 2000 dollars per capita).

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[24] GDP is denoted in thousands of constant 2000 dollars per capita. USA emissions are obtained from the World Resources Institute (2009). See Section A3.9 below for the values of the other stocks and flows in the year 2000.

A3.7. The calibration of the law of motion of the stock of knowledge

The yearly depreciation rate for knowledge commonly used is much lower than the one for capital (e.g. the Bank of Spain uses $\hat{\delta}^n = 0.15$, which would mean that knowledge dissipates almost entirely in one generation). We believe that the discount factor should be higher because of the intergenerational-public-good character of knowledge. A dollar invested in R&D by a firm may well generate no returns to the firm 25 years later, yet its impact to the accumulation of social knowledge capital may be substantial. Thus, as an approximation we take the depreciation rate of the stock of knowledge to be the same as that of physical capital, i. e., in generational terms, $\delta^n = \delta^k = 0.787$.

We approximate the year-to-year discounting with the same argument as in physical capital. If we denote by i^n the average annual expenditure per capita in knowledge, then we could write $(1-\delta^n)S^n_{t-1} + \frac{1-(1-\delta^n)^{25}}{1-(1-\delta^n)} i^n > S^n_t$. But, because investment in knowledge is written in efficiency units of labor per capita, then $\frac{1-(1-\delta^n)^{25}}{1-(1-\delta^n)} i^n_t = k_3 x^n_t$, that is, $k_3 = \frac{1-(1-\delta^n)^{25}}{1-(1-\delta^n)} \frac{i^n_t}{x^n_t}$, where $\frac{i^n_t}{x^n_t}$ is the wage of an efficiency unit of labor.

Now, we estimate $\frac{i^n_t}{x^n_t} = \frac{i^n_t}{\epsilon^n (1/3) x_t}$ where (1/3) x_t is the total efficient units of labor and ϵ^n the share of labor devoted to the production of knowledge. We take $\epsilon^n = 0.05$ (5% of total labor) and use the average values for the last generation (1976-2000) to obtain $\frac{i^n_t}{x^n_t} = \frac{i^n_{76-00}}{0.05(1/3)x_{76-2000}} = \frac{0.99}{0.02} =$ 49.5 thousands of 2000 dollars.

Hence, $k_3 = \frac{1-(1-\hat{\delta}^n)^{25}}{1-(1-\hat{\delta}^n)} \frac{i^n}{x^n} = 13.118 \times 49.5 = 649.34$.

The time series of the stock of knowledge is constructed by the perpetual inventory method, using US data for 1960-2000 and taking 1960 as initial value. For the initial stock of knowledge, $S^n_{1960} = \frac{i^n_{1960}}{\hat{\delta}^n + g^n} = \frac{0.421}{0.06+0.041} = 4.21$ thousands of constant 2000 dollars per capita, where i^n represents total expenditure per capita in R&D plus public and private investment in software,[25] and

[25] See Footnote 19 above.

g^n represents the average yearly growth rate between 1960-1970. The value for the stock of knowledge in the year 2000 is $\bar{S}^n_{2000}$ = 15.64 (in thousands of 2000 dollars per capita).

A3.8. The calibration of the education production function

The parameter k_4, capturing the productivity of education, plays an important role in the model. By definition, $k_4 = \frac{x_t}{x^e_{t-1}}$, where both the numerator and the denominator are measured in efficiency units. We can transform efficiency units into hours by the equality $\frac{x_t}{x^e_{t-1}} = \frac{(1+s)^T \hat{x}_t}{(1+s)^{T-1} \hat{x}^e_{t-1}} = (1+s)\frac{\hat{x}_t}{\hat{x}^e_{t-1}}$ (for some T), where $(1 + s)$ is the growth factor of human capital per generation, and where the "hats" represent annual data in hours. Hence, the calibration of k_4 is based on two rates: s and the share $\frac{\hat{x}^e_{t-1}}{\hat{x}_t}$ of time devoted to education out of total time. Note that k_4 is increasing in s and decreasing in the share $\frac{\hat{x}^e_{t-1}}{\hat{x}_t}$.

We take the average yearly growth rate $\hat{s}$ of the human capital stock a value $\hat{s} = 0.67\%$, which yields the per-generation factor $(1 + s) = (1 + \hat{s})^{25} = 1.0067^{25}$. This figure, supported by the 1960-85 average provided by de la Fuente and Domènech (2001), is lower than the 0.93% average for 1960-2000 found in Robert Barro and Jong-Wa Lee (2001).

The rate $\frac{\hat{x}^e_{t-1}}{\hat{x}_t}$ is the product of the rate of education in labor and the rate of labor in total time. From our time series, we infer that about 10% of total labor is devoted to education, and that labor accounts for 1/3 of total time. These figures are conservative in the sense that lower values for them would yield a higher value for k_4. In summary, we take

$$k_4 = (1.0067)^{25} \frac{1}{\frac{0.1}{3}} == 35.45 .$$

A3.9. Initial values in the benchmark year 2000

The values for the stock of physical capital, $\bar{S}^k_{2000}$ = 73.65, and knowledge, $\bar{S}^n_{2000}$ = 15.64 (in thousands of 2000 dollars per capita), are obtained by using the perpetual inventory method as reported in sections A3.6-7 above.

We take $\bar{S}^m_{2000}$ = 177.1 GtC (or 83 ppm) as the year 2000 atmospheric CO_2 concentration above pre-industrial level (of approximately 595.5GtC in 1850) from the CAIT Indicator Framework Paper (World Resources Institute, WRI, 2009). As for emissions, we take $\bar{e}_{2000}$ = 6.58 GtC also from the World Resources Institute (2009) as the world annual CO_2 emissions from energy (fossil fuels and cement) in GtC.[26]

The series of human capital stock (in efficiency units) is constructed normalizing year 1950 equal to 1 and taking the average yearly growth rate of human capital stock equal to 0.67% (de la Fuente and Domènech, 2001). Hence, $x_t = 1.0067^{t-1950}$ in 1950-efficiency units, and therefore $\bar{x}_{2000}$ = 1.0067^{50} = 1.396.

We take education to occupy 10% of labor time. And consequently, $\bar{x}^e_{2000}$ = 1.396×1/3×0.1 = 0.0465 in 1950-efficiency units.

Finally, see the calibration of the production functions in Section A3.5 above for total income, consumption and investment.

APPENDIX 4. NORDHAUS'S SOCIAL WELFARE FUNCTION AND THE CALIBRATION OF ITS PARAMETERS

A4.1. A long-lived consumer

The traditional theory of economic growth considers the accumulation of physical capital, in particular the tradeoff between present consumption and the enhanced consumption possibilities of future generations offered by saving. It often postulates a long-lived representative consumer, whose preferences are representable in an additively separable manner as the discounted sum of future single-date subutilities, one for each future date. If only the consumption c_t at each date enters the single-date subutility function, and if such function is of the form $\frac{1}{1-\eta}c^{1-\eta}$, then the consumer's preferences are represented by the utility function

$$\sum_{t=1}^{T}\frac{1}{1-\eta}c^{1-\eta}\frac{1}{(1+\delta)^t}\ , \qquad \text{(A4.1)}$$

[26] Once we include CO_2 emissions from land use change (7.62 $GtCO_2$) and from other Kyoto gases (9.72 $GtCO_2e$), our value (41.42 $GtCO_2e$) is consistent with the 42 GtCO2e total GHG emissions in 2000 reported in the Stern Review (page 170).

where $T \leq \infty$, $\eta > 0$ (for $\eta = 1$, ln c replaces $\frac{1}{1-\eta}c^{1-\eta}$), and where the discount factor $\frac{1}{1+\delta}$ (or the discount rate δ) reflects the consumer's marginal rate of intertemporal substitution: a more impatient consumer has a larger δ, and attaches little value to a unit of consumption made available to him far into the future.

A4.2. Nordhaus's social welfare function

The social welfare function in Nordhaus (1991, 1994, 2008a,b), and Nordhaus and Boyer (2000), see (2) in Section 6.1 above, is similar to (A4.1), but with a quite different meaning. Now t = 1, 2, … represent generations, and c_t is the consumption per capita of Generation t. As noted in Section 6.1 above, Nordhaus's (2008a) calls δ and η "central" and "unobserved normative parameters," affecting "the relative importance of the different generations." The parameter δ is a "pure social time discount rate:" a high δ means that the welfare of a generation born far into the future counts very little in the social welfare function. The second one represents "the aversion to inequality of different generations." Informally speaking, if the rates of growth turn out to be negative, then δ and η push in opposite directions, a high δ favoring the earlier generations and a high η favoring the later, less well off, generations. But for positive rates of growth, when the latter generations are better off, high values of either δ or η favor the earlier generations. This is the case in the paths proposed by Nordhaus (2008a, b).

A4.3. Nordhaus's calibration of the parameters

Nordhaus (2008a,b) calibrates η and $\hat{\delta}$ as follows. First, he adopts the "Ramsey equation"

$$\hat{r} = \hat{\delta} + \eta\,\hat{g}\,, \tag{A4.2}$$

where $\hat{r}$ is the real per year rate of interest on capital and $\hat{g}$ is the per year rate of growth of consumption. Nordhaus (2008a, p. 60-61) justifies equation (A4.2) by the maximization of the function (A4.1) subject to some constraints. In his words, and noting that his symbol ρ (resp. α) corresponds to the δ (resp. η) of the present paper:

> "The basic economics can be described briefly. Assume a time discount rate of ρ and a consumption elasticity of α. Next, maximize the social welfare function described earlier and in the Appendix with a constant population and a constant

> rate of growth per generation g^*. This yields the standard equation for the equilibrium real return on capital, r^*, given by $r^* = \rho + \alpha g^*$."

Second, he infers $\hat{r}$ and $\hat{g}$ from "observed economic outcomes as reflected by interest rates and rates of return on capital" (p. 33-34).

Third, he chooses $\hat{\delta}$ and η subject to the Ramsey equation, which gives one degree of freedom. In particular, Nordhaus (2008a, p. 178) takes the values $(\hat{r}, \hat{g}) = (0.055, 0.02)$. Equation (A4.2) then holds for any $(\hat{\delta}, \eta)$ pair satisfying $\hat{\delta} = 0.055 - 0.02\eta$, in particular by the values $(\hat{\delta}, \eta) =$ (0.015, 2) chosen by Nordhaus (2008a).[27]

Summarizing, equation (A4.2) is obtained by the constrained maximization of (A4.1), whereas $\hat{r}$ and $\hat{g}$ are deduced from observed behavior. Inserting $\hat{r}$ and $\hat{g}$ into (A5.2) could make sense if, as in Section A4.1 above, observed behavior was generated by a single long-lived consumer who solves the optimization program. But in this case the parameters $(\hat{\delta}, \eta)$ would be "positive," rather than "normative," whereas Nordhaus's analysis concerns a world of many distinct generations, with parameters $(\hat{\delta}, \eta)$ which are "normative." It is peculiar to think of rates of return observed in the market as depending on these "normative" parameters, in particular on the aversion, by past and current market participants, to inequality among generations.

In addition, because Nordhaus (2008a) gives little detail on the constraints of the optimization program leading to (A4.2), it is hard to evaluate the assumption that r^* and g^* are constant at the solution. In any event, the solution paths will depend on the initial conditions on the stocks, so that the constancy of rates can typically be justified only asymptotically.[28]

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[27] Elsewhere in the book he refers to a $\hat{r}$ of 0.04 (pp. 9-11) and to a $\hat{g}$ of 0.013 (p. 108).

[28] Consider, for instance, the traditional Ramsey problem, with capital but without environmental stocks: An infinitely lived consumer maximizes $\int_0^\infty \frac{1}{1-\eta} c(t)^{1-\eta} e^{-\delta t} dt$ subject to the law of motion of capital k_R and the initial condition $k_R(0) = k_0$. Let capital depreciate at the rate δ_R, and let the production function be $Ak_R{}^{\psi} e^{nt}$, where $\psi \in (0,1)$ and $n \geq 0$ is the rate of exogenous technological change. The constraint is then $\dot{k}_R(t) \leq Ak_R(t)^{\psi} e^{nt} - c(t) - \delta_R k_R(t)$. Writing the Hamiltonian as $H(c, k_R, \lambda) = (1-\eta)^{-1} c^{1-\eta} e^{-\delta t} + \lambda[Ak_R{}^{\psi} e^{nt} - c - \delta_R k_R]$, at the solution path one must have (see, e. g., George Hadley and Murray Kemp, 1971, Th. 4.3.1) (a) $\frac{\partial H}{\partial c} = 0$, i. e., $c^{-\eta} e^{-\delta t} - \lambda = 0$, and (b) $-\frac{\partial H}{\partial k_R} = \dot{\lambda}$, i. e., $-\lambda(A\psi k_R{}^{\psi-1} e^{nt} - \delta_R) = \dot{\lambda}$. From (a), $-\eta c^{-\eta-1} \cdot \dot{c} \cdot e^{-\delta t} + c^{-\eta} \cdot e^{-\delta t} .(-\delta) = \dot{\lambda}$, which together

A4.4. Discounted utilitarianism

The parameter η could also be interpreted, following the classical utilitarians, as an index of the concavity of a common, cardinal and interpersonally unit-comparable utility function displaying decreasing marginal utility.[29] The function (2) would then be the social welfare function of discounted utilitarianism. But we find discounted utilitarianism ethically unacceptable, at least for the high (pure time) discount rates δ typically used in the literature, which put a weight on the utility of future generations much lower than that of the present generation. The only ethical justification for putting a lower weight on the welfare of future generations in the utilitarian calculus should be based on a positive probability of extinction of mankind. As argued in the Stern Review, this rationale would perhaps support a discount rate of $\hat{\delta}$ = 0.001 = 0.1% *per annum*, associated with a 0.905 probability of mankind's surviving 100 years. Of course, a rigorous development of this idea requires an explicit model of uncertainty: see Section 5 above and Llavador *et al.* (2009), where the problem is formulated as one of an impartial observer with von Neumann-Morgenstern preferences over uncertain future worlds.

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with (b), using (a) again and dividing through by $c^{-\eta}\cdot e^{-\delta t}$, gives $A\psi k_R{}^{\psi-1}e^{nt}-\delta_R=\delta+\eta\cdot\frac{\dot{c}}{c}$, a time-dependent form of (A5.2). Assume now that $\frac{\dot{c}}{c}=\bar{g}$, a constant, i. e., $c(t)=c_0e^{\bar{g}t}$ for some $c_0>0$. The last equation then reads $A\psi k_R(t)^{\psi-1}e^{nt}=\delta_R+\delta+\eta\bar{g}$, i. e., $k_R(t)=(\delta_R+\delta+\eta\bar{g})^{\frac{1}{\psi-1}}(A\psi)^{\frac{1}{1-\psi}}e^{\frac{n}{1-\psi}t}$, and the initial condition $k_R(0)=k_0$ requires $k_0=(\delta_R+\delta+\eta\bar{g})^{\frac{1}{\psi-1}}(A\psi)^{\frac{1}{1-\psi}}$. Writing $k_R(t)=k_0e^{\frac{n}{1-\psi}t}$ and dividing through by k_R, the law of motion becomes $\frac{n}{1-\psi}=Ak^{\psi-1}e^{nt}-\frac{c}{k}-\delta_R$, i. e., $\frac{n}{1-\psi}=A[k_0e^{\frac{n}{1-\psi}t}]^{\psi-1}e^{nt}-\frac{c_0}{k_0}e^{\bar{g}t}e^{-\frac{n}{1-\psi}t}-\delta_R$, or $\frac{n}{1-\psi}=Ak_0{}^{\psi-1}-\frac{c_0}{k_0}e^{\bar{g}t}e^{-\frac{n}{1-\psi}t}-\delta_R$, $\forall t$, which implies that $\bar{g}=\frac{n}{1-\psi}$. But then the parameters $(\eta,\delta,A,\psi,n,\delta_R,k_0)$ must belong to the set of measure zero defined by the equality $k_0=(\delta_R+\delta+\eta\frac{n}{1-\psi})^{\frac{1}{\psi-1}}(A\psi)^{\frac{1}{1-\psi}}$.

[29] See Roemer (1998) for definitions.

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